A slice of Catalan combinatorics from the perspective of Coxeter sortable elements of the symmetric group

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Summary

1. Coxeter sortable elements of the symmetric group
   - Symmetric group
   - Coxeter sortable elements
2. Non crossing partitions
   - Geometric non crossing partitions
   - Bijection with the c-sortable elements
3. Binary trees
   - Binary trees and SSA algorithm
   - Link with c-sortable elements

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   - Symmetric group
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3. Binary trees

Notations

Let $n \in \mathbb{N}^*$. We note $\mathfrak{S}_n$ the symmetric group of order n.

An element $\sigma \in \mathfrak{S}_n$ is represented by its one line notation : $\sigma(1)\sigma(2)\dots\sigma(n)$.

Example : $\mathfrak{S}_3 = \{1\,2\,3,\ 2\,1\,3,\ 2\,3\,1,\ 1\,3\,2,\ 3\,1\,2,\ 3\,2\,1\}$.

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Example : $\mathfrak{S}_3 = \{1\,2\,3,\ 2\,1\,3,\ 2\,3\,1,\ 1\,3\,2,\ 3\,1\,2,\ 3\,2\,1\}$.

Coxeter group : $\mathfrak{S}_n \simeq \left\langle s_1, \dots, s_{n-1} \;\middle|\; \begin{array}{ll} s_i^2 = 1 & \forall 1 \leq i \leq n-1 \\ s_i s_j = s_j s_i & \forall |i-j| > 1 \\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} & \forall 1 \leq i \leq n-2 \end{array} \right\rangle$

COXETER elements

A **COXETER element** of a COXETER group W is an element of W that is the product of all the generators exactly once.

Examples : $s_1 s_2 \dots s_{n-1},\ s_1 s_3 \dots s_{n-1} \dots s_4 s_2.$

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A COXETER element of $\mathfrak{S}_n$ is permutation c of $\mathfrak{S}_n$ that is a great cycle of the form $(1, a_1, \dots, a_k, n, b_l, \dots, b_1)$ where $k + l = n - 2$, $a_1 < \dots < a_k$ and $b_1 < \dots < b_l$.

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It is characterized by a partition of $\{2, \dots, n-1\} = \underbrace{\{a_1, \dots, a_k\}}_{L_c} \sqcup \underbrace{\{b_1, \dots, b_l\}}_{R_c}$.

Examples : let $n = 6$, $(1, 3, 4, 6, 5, 2)$, $(1, 6, 5, 4, 3, 2)$, $(1, 2, 3, 4, 5, 6)$, ~~$(1, 5, 2, 6, 3, 4)$~~ .

COXETER sortable elements

Let c be a COXETER element of $\mathfrak{S}_n$. A permutation $\sigma \in \mathfrak{S}_n$ is c-sortable [Rea05] if its one line notation avoids the following patterns :

- $ki \dots j$ for $i < j < k$ and $j \in L_c$
- $j \dots ki$ for $i < j < k$ and $j \in R_c$.

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Examples : let $n = 7$ and $c = (1, 3, 4, 6, 7, 5, 2)$.

- 7416325 is not c-sortable because it contains a pattern $ki\dots j$ with $j = 6 \in L_c$.
- 6521743 is not c-sortable because it contains a pattern $j\dots ki$ with $j = 5 \in R_c$.
- 3167425 is c-sortable because it avoids all the patterns.

CATALAN

Let $n \in \mathbb{N}^*$. For any COXETER element c of $\mathfrak{S}_n$, there are $C_n = \frac{1}{n+1}\binom{2n}{n}$ c-sortable elements.

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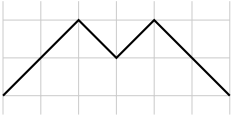
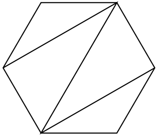
This means the c-sortable elements are in bijection with all the objects enumerated by the Catalan numbers.

Examples :

Dyck paths of length $2n$



Triangulations of $(n+2)$-gons



Well parenthesized expressions of $n+1$ factors

$$((a(bc))d)(ef)$$

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2. Non crossing partitions

- Geometric non crossing partitions
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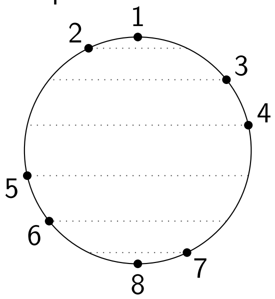
Labeling the circle

Let $n \in \mathbb{N}^*$ and place on a circle the numbers from 1 to n such that :

- 1 is at the highest point and n is at the lowest point,
- no two numbers are on the same height,
- reading from top to bottom the numbers are increasing.

Example : $n = 8$



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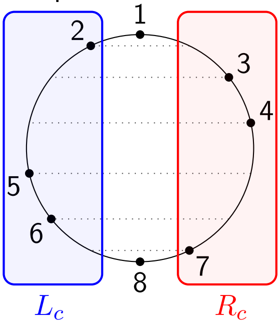


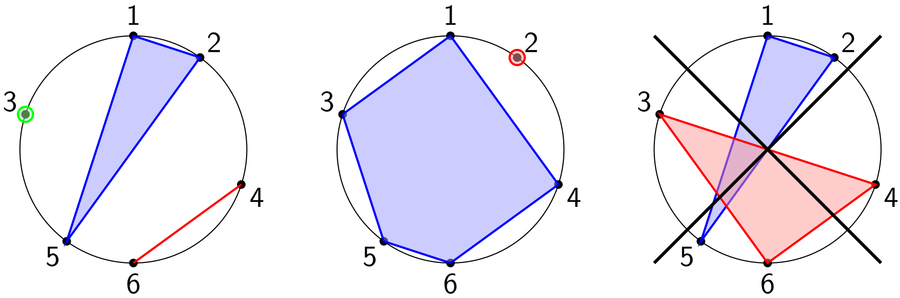
$c = (1, 2, 5, 6, 8, 7, 4, 3)$

A c-labeling is a labeling such that on the left are the elements of L_c and on the right the elements of R_c

Non crossing partitions

A c-non crossing partition is a set of non crossing polygons with vertices the marked points of a c-labeled circle. Single points and segments are considered as polygons.

Examples : $n = 6$ and $c = (1, 3, 5, 6, 4, 2)$



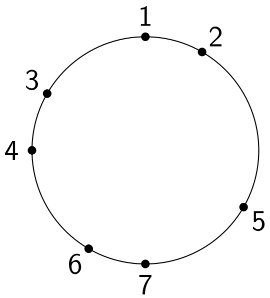
Bijection with the c-sortable elements

Theorem (READING [Rea05])

The set of c-sortable elements is in bijection with the set of c-non crossing partitions via an explicit map called nc_c.

Example : let $n = 7$, $c = (1, 3, 4, 6, 7, 5, 2)$ and $\sigma = 3167425$. It is a c-sortable element.

$\sigma = 3167425 \longrightarrow$



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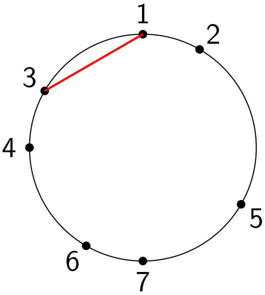

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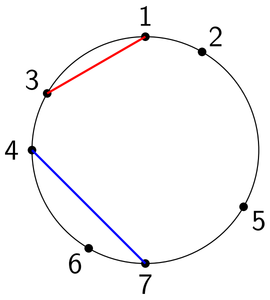

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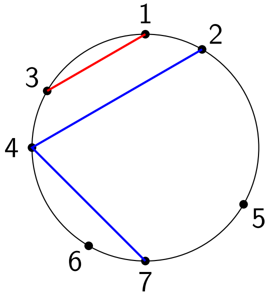

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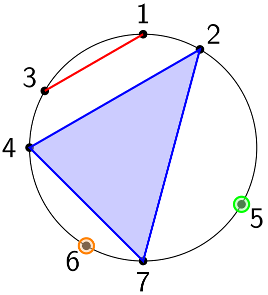
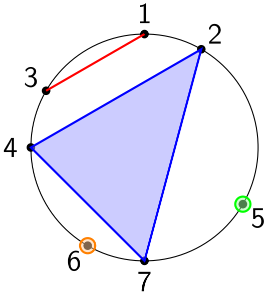


$= nc_c(\sigma)$.

Inverse map

The inverse map of nc_c can be computed by selecting the polygons of a c-non crossing partition in a specific order [Gob18].

Example : Let's use the c-non crossing partition we computed in the previous slide.



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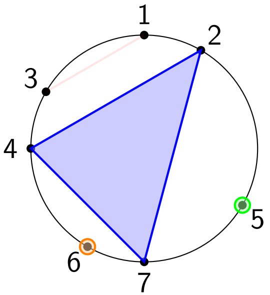

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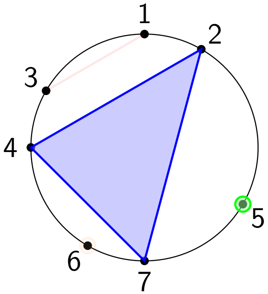

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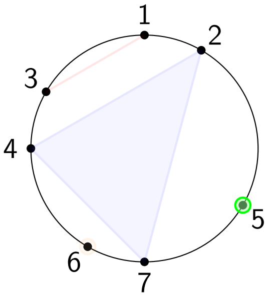

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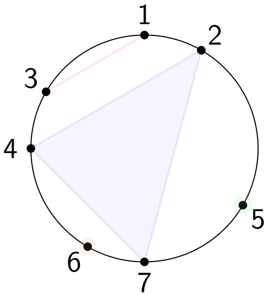

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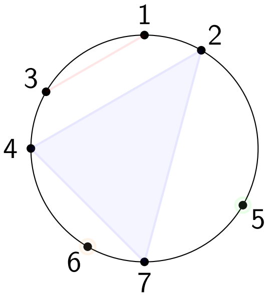

$\longrightarrow \{3,1\},\{6\},\{7,4,2\},\{5\} \longrightarrow 3167425 = \sigma$

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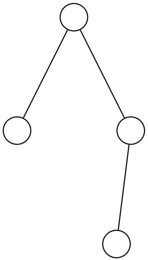
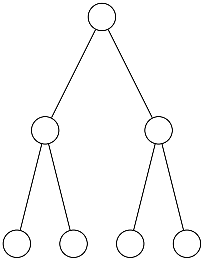
Binary trees

A binary tree is either an empty tree or a node with exactly one left child and one right child that are binary trees. The size of a binary tree is the number of nodes in the tree.

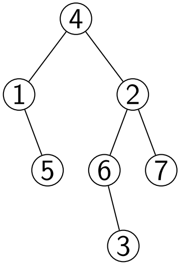
Examples :



A binary tree.



A complete binary tree.



A labeled binary tree.

Binary search trees, descending trees

A binary search tree is a labeled binary tree such that the label of each node is larger than the labels of its left child and smaller than the labels of its right child.

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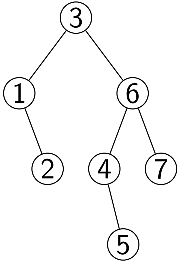


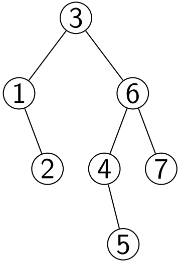
A binary search tree

Binary search trees, descending trees

A binary search tree is a labeled binary tree such that the label of each node is larger than the labels of its left child and smaller than the labels of its right child.

A descending tree is such that the label of each node is larger than the labels of its descendants.

Examples :



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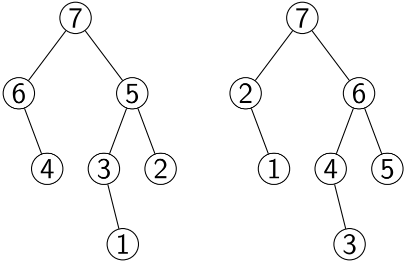

Two descending trees of the same shape

SSA algorithm

Theorem (SSA algorithm [HNT04])

There is an explicit bijection

$$\mathfrak{S}_n \simeq \left\{ (T,Q) \;\middle|\; \begin{array}{l} T \text{ is a binary search tree of size } n \text{ and} \\ Q \text{ is a descending tree of the same shape as } T \end{array} \right\}$$

Example : Let $n = 7$ and $\sigma = 2154763$.

$T(\sigma) =$ $\qquad$ $Q(\sigma) =$

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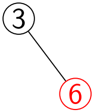

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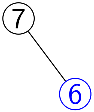

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Example : Let $n = 7$ and $\sigma = 2154763$. (position $= 5$)

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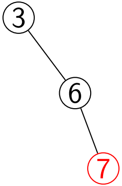

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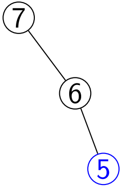

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Example : Let $n = 7$ and $\sigma = 2154763$. (position $= 4$)

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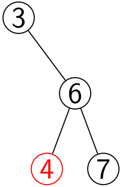

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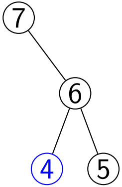

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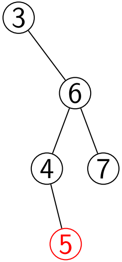
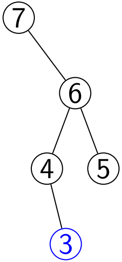


$Q(\sigma) =$



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Example : Let $n = 7$ and $\sigma = 2{\color{red}1}54763$. (position $= {\color{blue}2}$)

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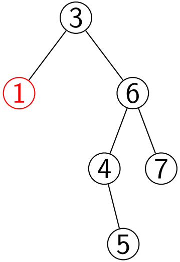

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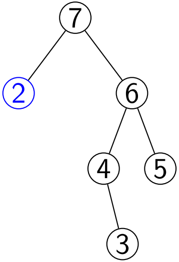

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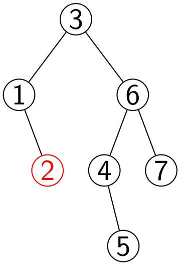
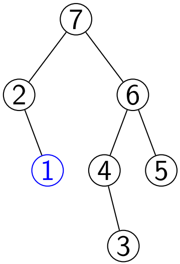
$T(\sigma) =$ 

$Q(\sigma) =$ 

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Example : Let $n = 7$ and $\sigma = 2154763$.

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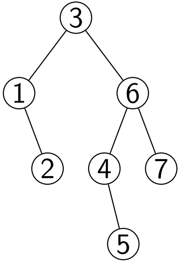
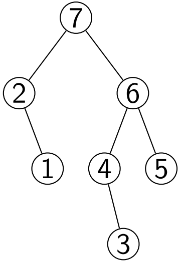


$Q(\sigma) =$



Sylvester congruence

SSA algorithm : $\sigma \in \mathfrak{S}_n$ is encoded by $(T(\sigma), Q(\sigma))$.

What happens if we forget $Q(\sigma)$? Can we describe all $\sigma' \in \mathfrak{S}_n$ s.t. $T(\sigma) = T(\sigma')$?

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Yes! $T(\sigma) = T(\sigma')$ iff σ' can be obtained from σ by a series of transformations of the form $ki \dots j \leftrightarrow ik \dots j$ with $i < j < k$ $\longrightarrow$ Sylvester congruence on $\mathfrak{S}_n$.

Example : $\sigma' = 5421763$ has the same binary search tree than $\sigma = 2154763$

$$2154763 \rightarrow 2514763 \rightarrow 2541763 \rightarrow 5241763 \rightarrow 5421763$$

Link with c-sortable elements

If $c = (1, 2, 3, \ldots, n-1, n)$, then we have the following one to one maps :

2143

σ with no $ki \ldots j$ pattern
$=$ a c-sortable element

Link with c-sortable elements

If $c = (1, 2, 3, \ldots, n-1, n)$, then we have the following one to one maps :

$$2143 \qquad \leftrightarrow \qquad \left\{ \begin{array}{c} 2143 \\ 2413 \\ 4213 \end{array} \right\}$$

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A sylvester class

Link with c-sortable elements

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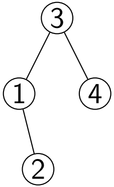

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Link with c-sortable elements

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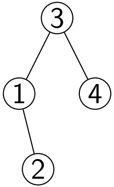
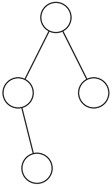
$2143 \leftrightarrow \left\{ \begin{array}{c} 2143 \\ 2413 \\ 4213 \end{array} \right\} \leftrightarrow$  $\leftrightarrow$ 

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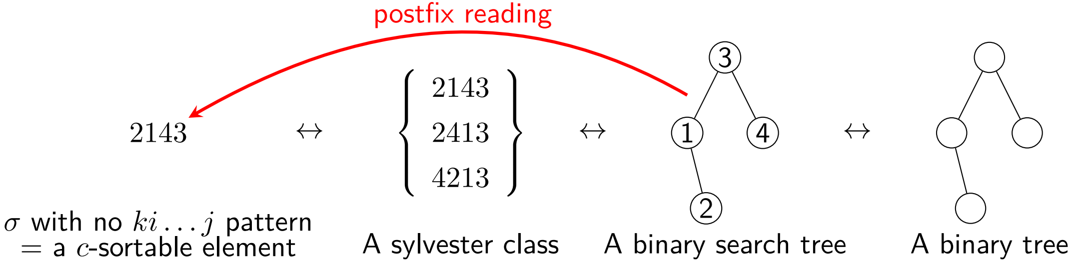

The map from binary trees to c-sortable elements can be directly obtained with a postfix reading of the associated binary search tree.

What I do

Reading's map $nc_c : \{c\text{-sortable elements}\} \to \{c\text{-non crossing partitions}\}$ is well defined for any (finite rank) Coxeter group, and is a bijection in all finite Coxeter groups.

In infinite Coxeter groups, it is only injective, but never surjective. For example, in type $\widetilde{A}_1$ and $c = st$, all reflections are c-non crossing but all the following ones are not in the image of nc_c : tst, $tstst$, $tststst$, ...

My goal is to define a generalized notion of c-sortable elements, at least in the affine types, such that Reading's map can be naturally extended to be a bijection.

Thank you for your attention!