# The ascent order on Dyck paths

with Jean-Luc Baril, Sergey Kirgizov (Dijon, F) and Mehdi Naima (Paris, F)

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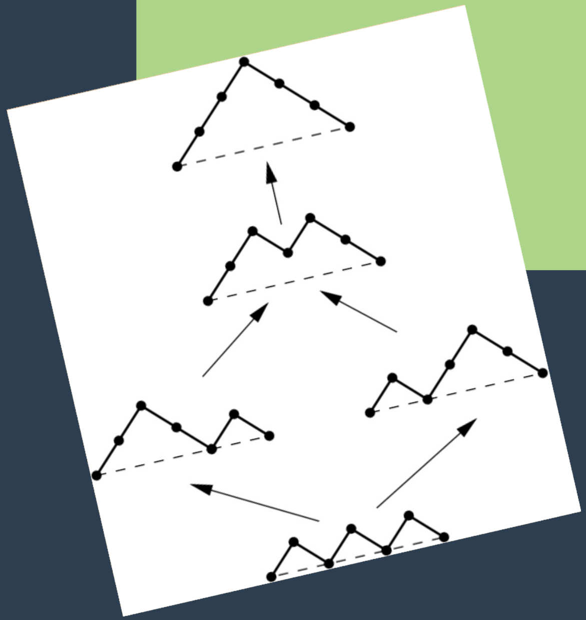
arXiv:2409.15982

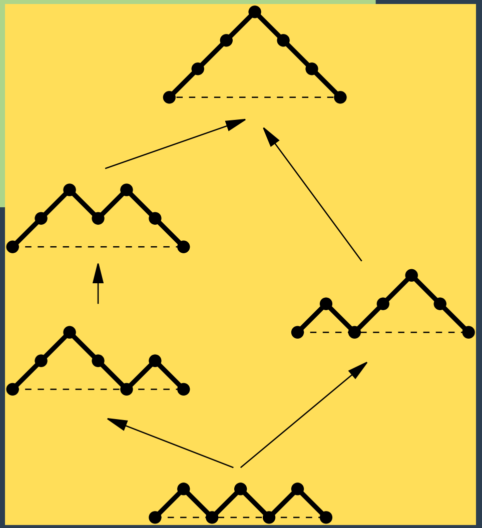
# Outline

- A new family of **lattices** $A_n$
- **Interval** counting
- Interesting **subposets** and their intervals
- Connection with **sylvester congruence classes**

[Hivert, Novelli, Thibon 05]

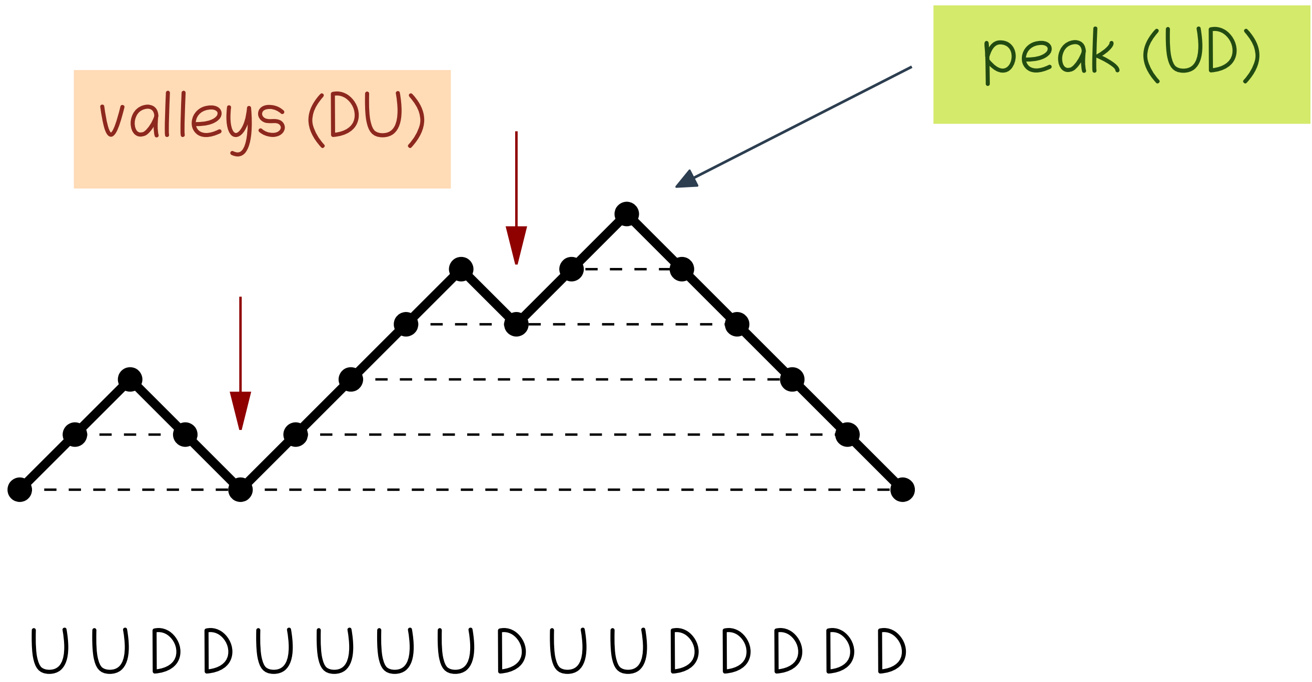
# I. Two orders on Dyck paths



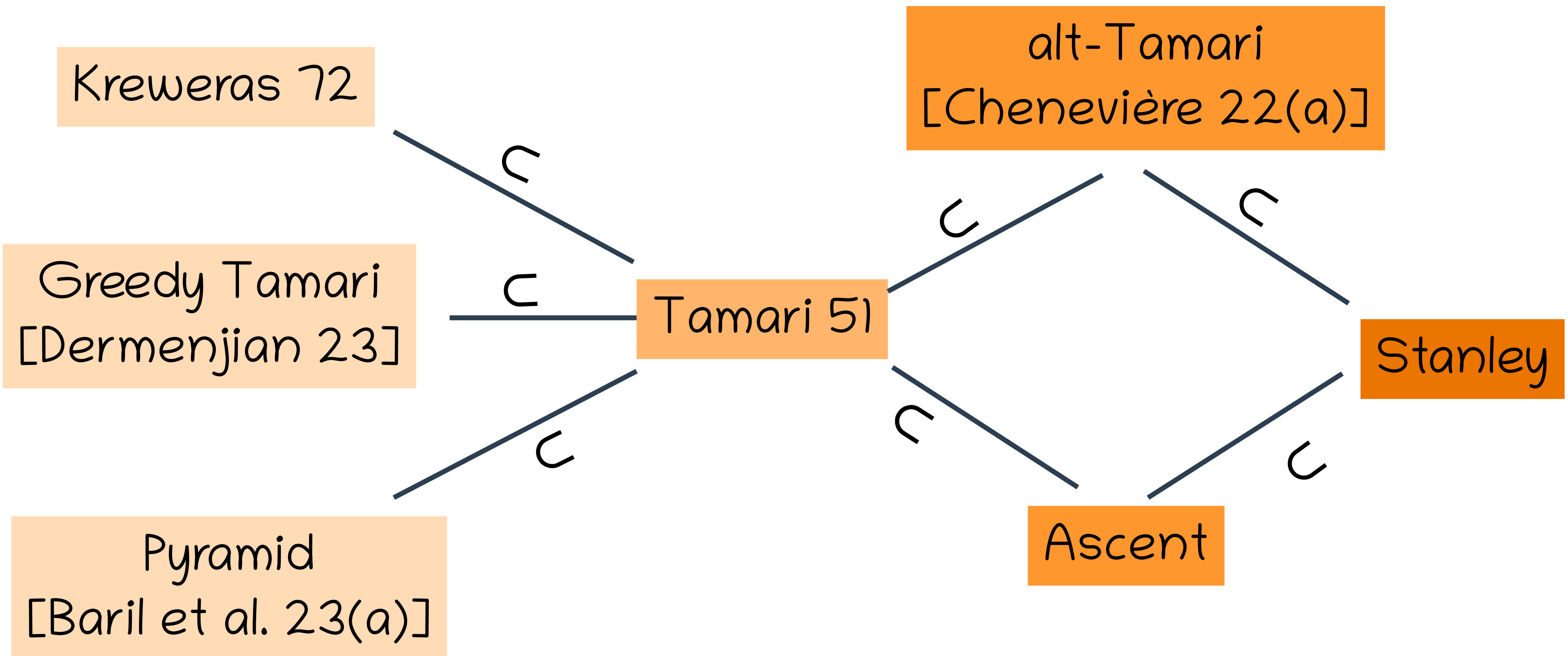


# Dyck paths

- A Dyck path of size n=8 (size=number of up steps)



# Many posets on Dyck paths of size n

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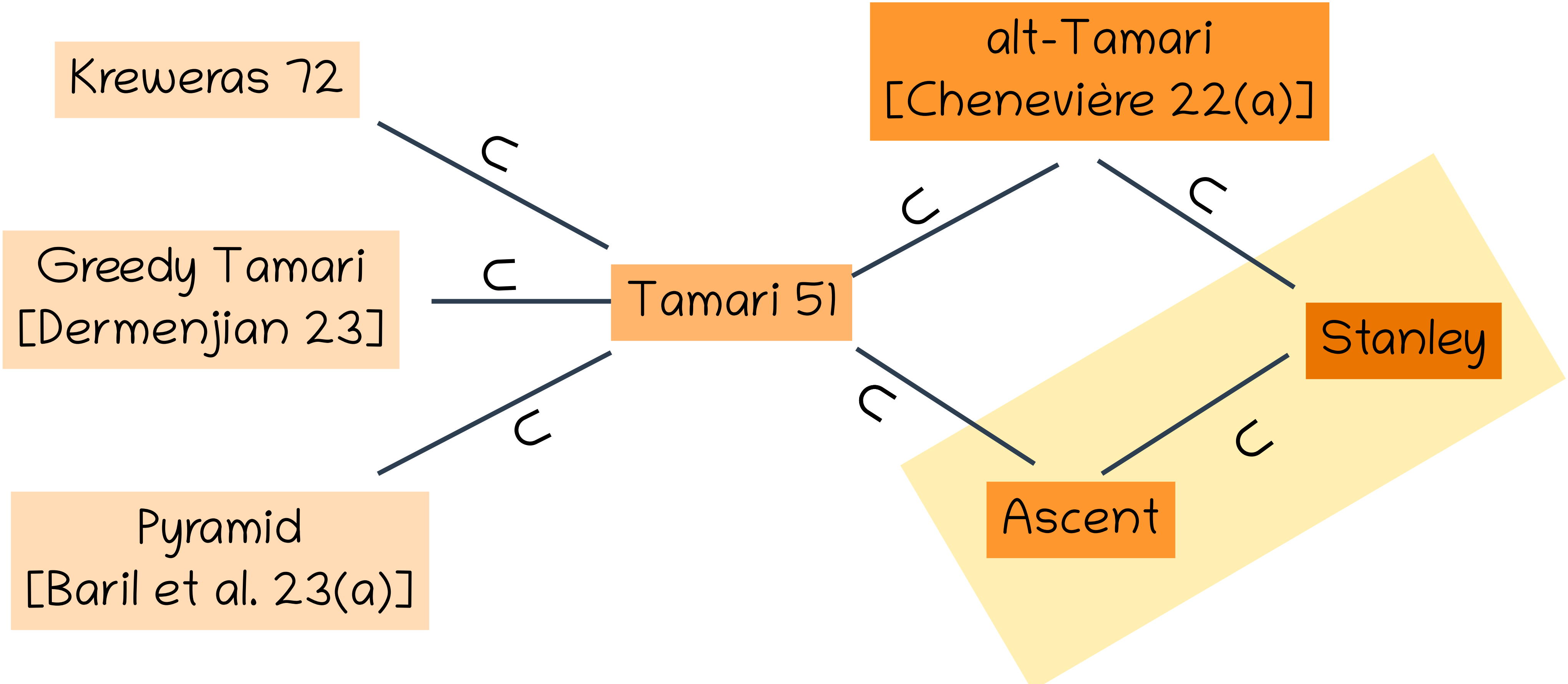


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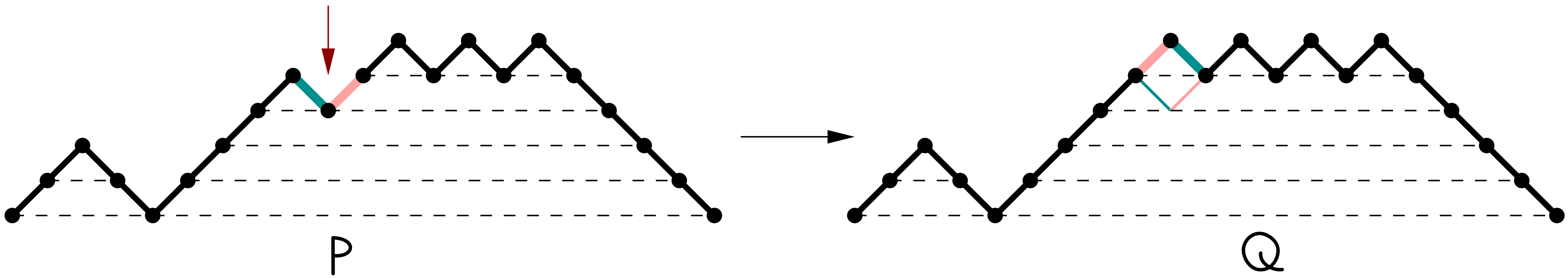

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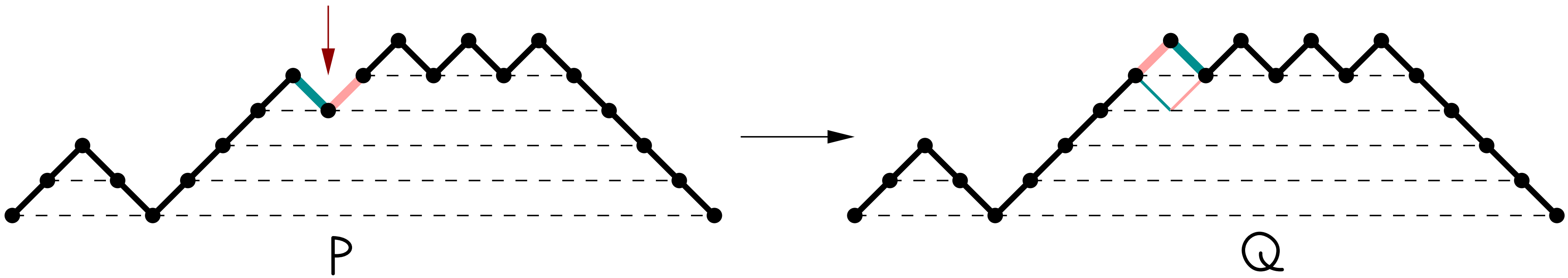

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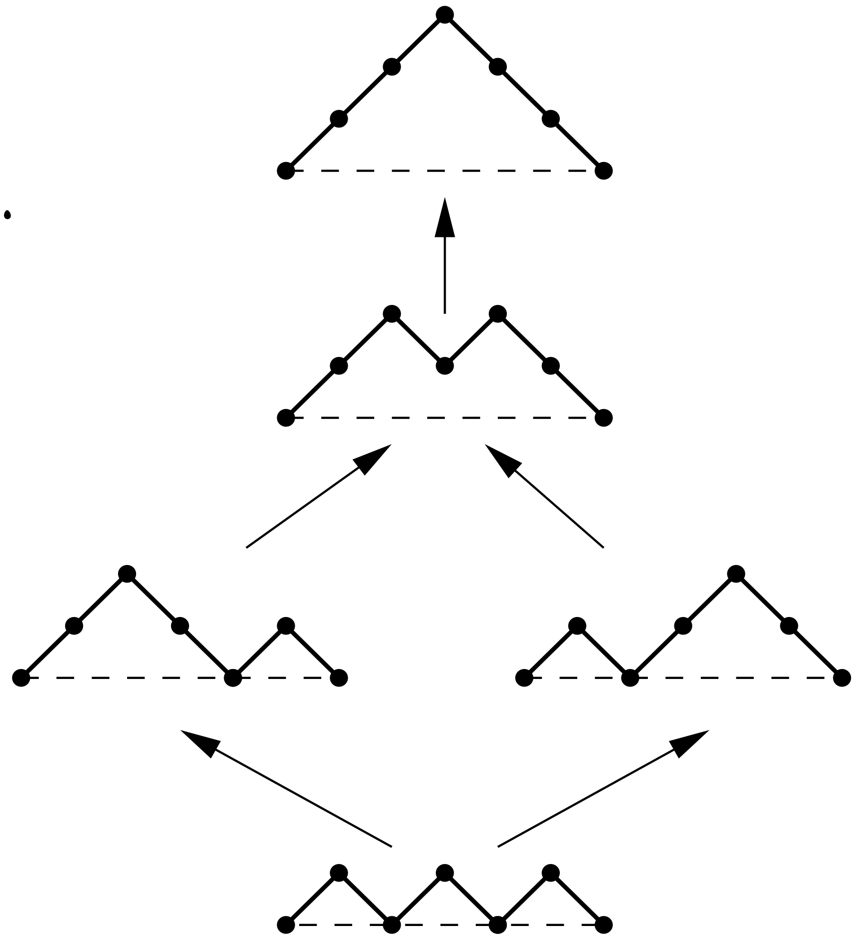

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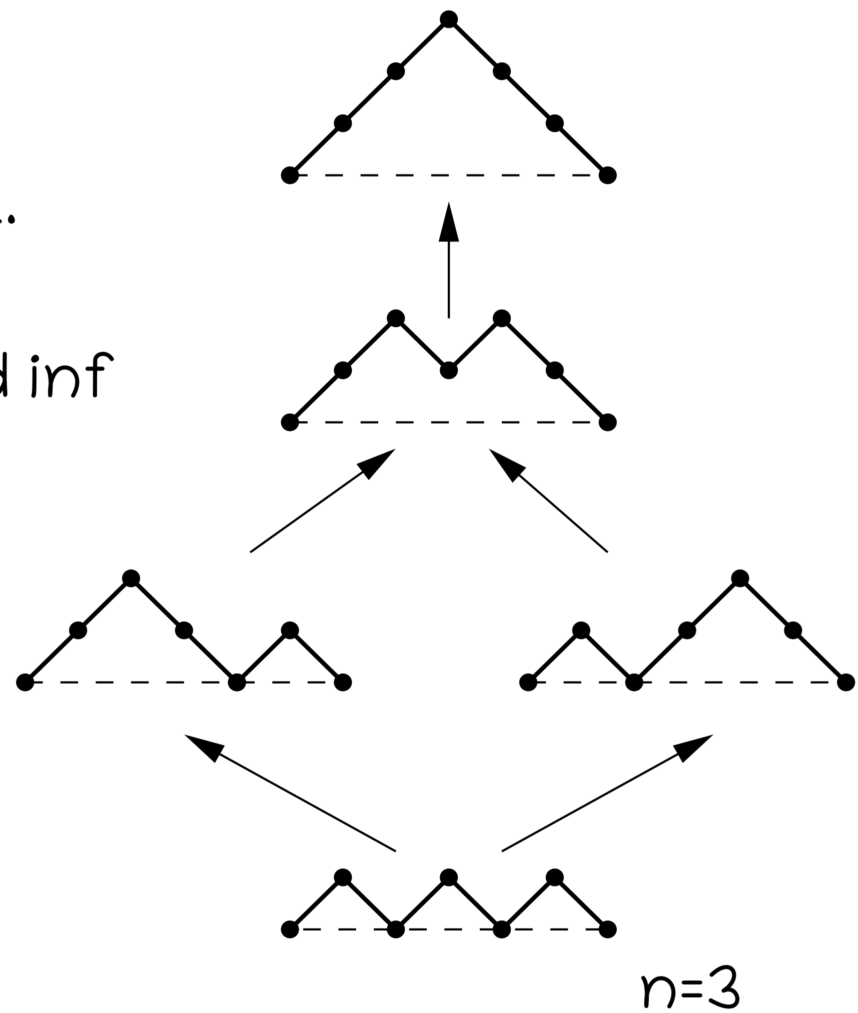
- **Lattice** structure: existence of sup and inf



# The ascent poset (or: greedy Stanley lattice?)

- A poset on Dyck paths with n up steps
- **Cover relations:** choose a valley in the path P.

  Swap the down step and the ascent that follows

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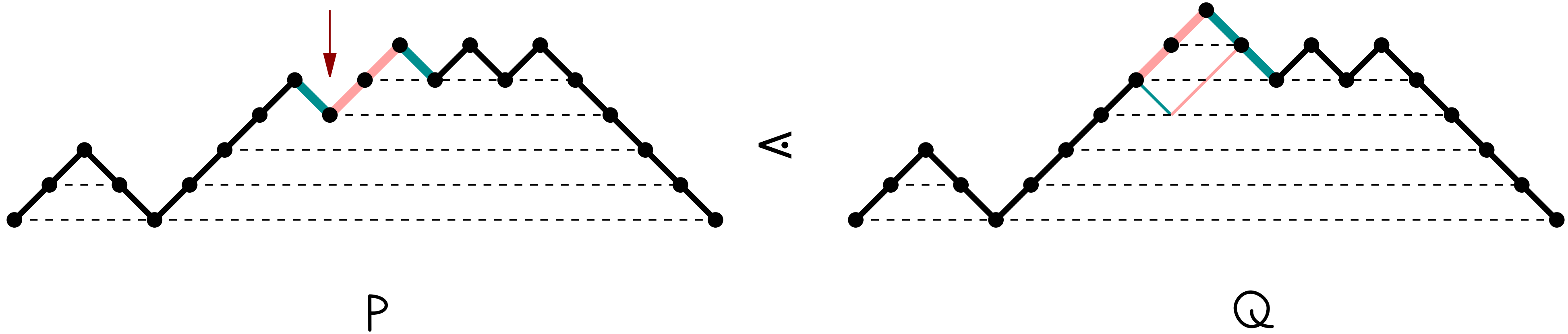
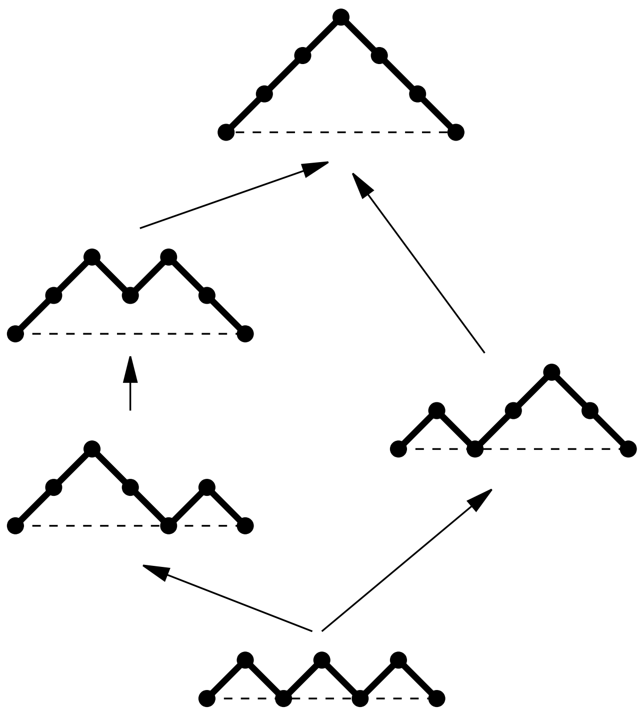



[Chenevière, Nadeau...]

# Ascent posets: n = 3, 4



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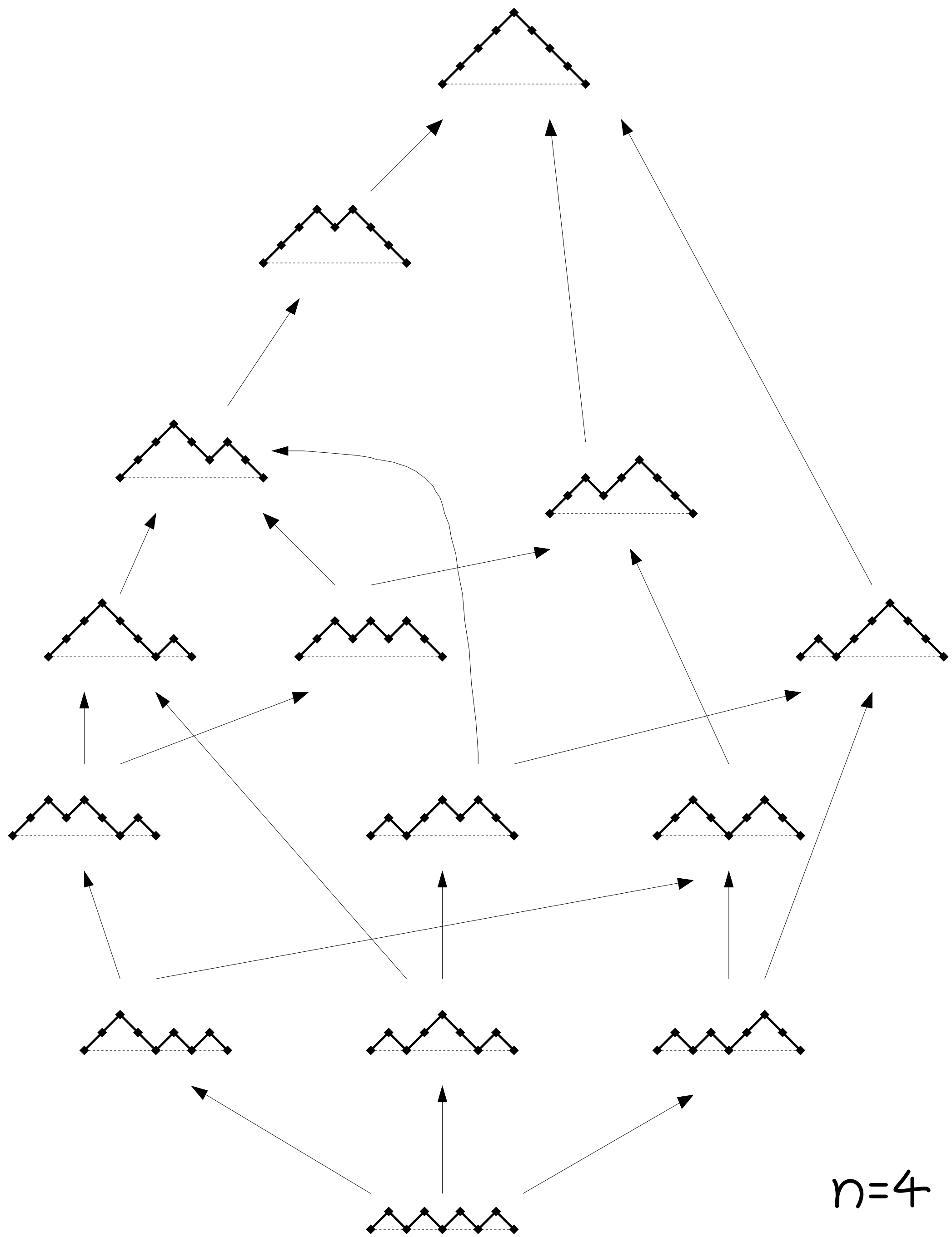
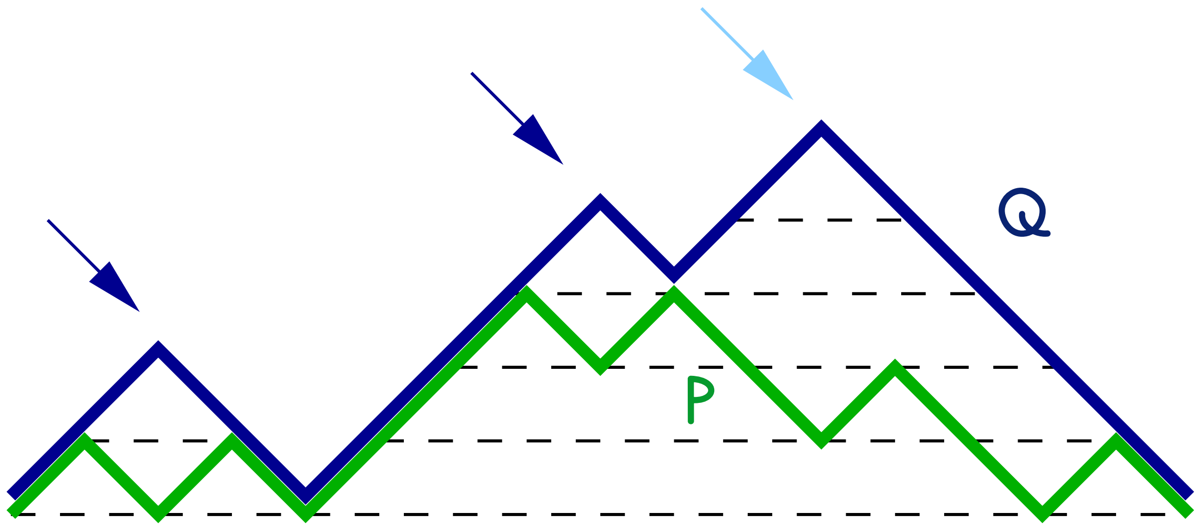


n=4

# A characterization of the ascent order

**Proposition.** In the ascent poset, $P \preccurlyeq Q$ iff

- P lies below Q
- every descent of Q **is** (i.e. lies on the same diagonal as) a descent of P.



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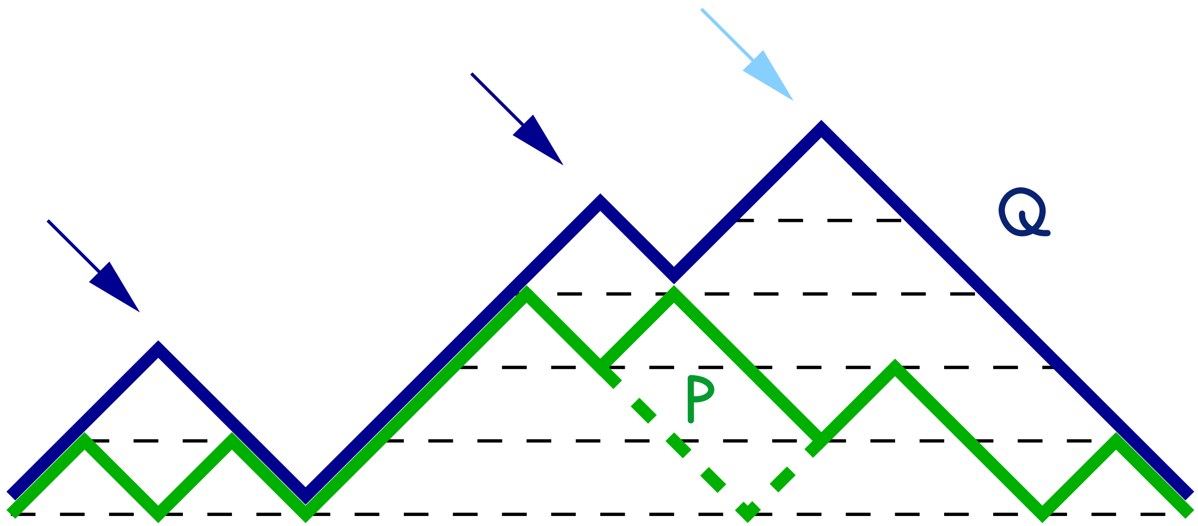

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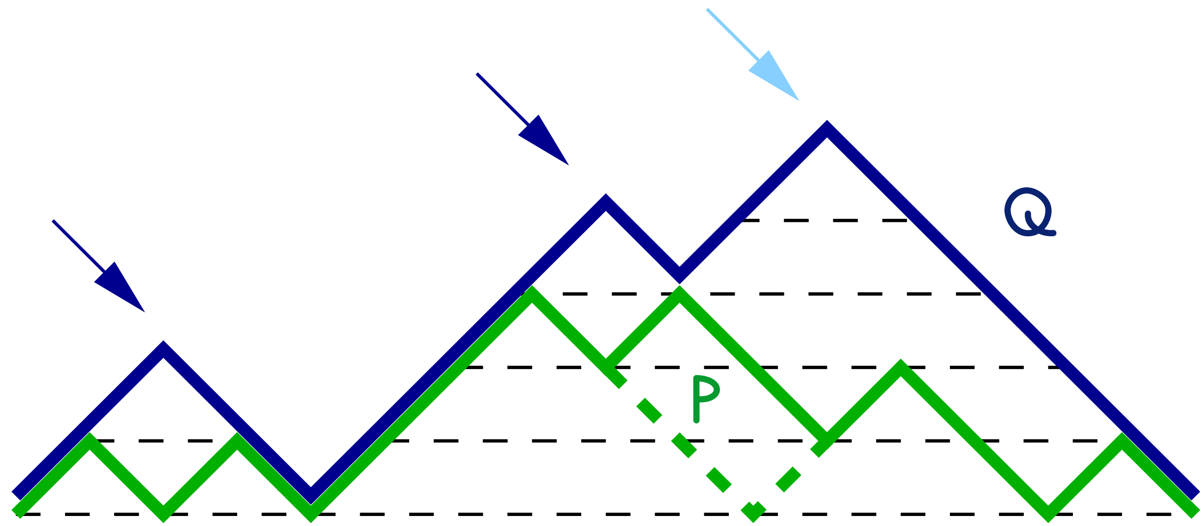


**Applications:**

- **lattice** structure
- recursive construction of **intervals**

# II. The number of intervals

Interval $[P,Q] \sim (P,Q)$ with $P \preccurlyeq Q$

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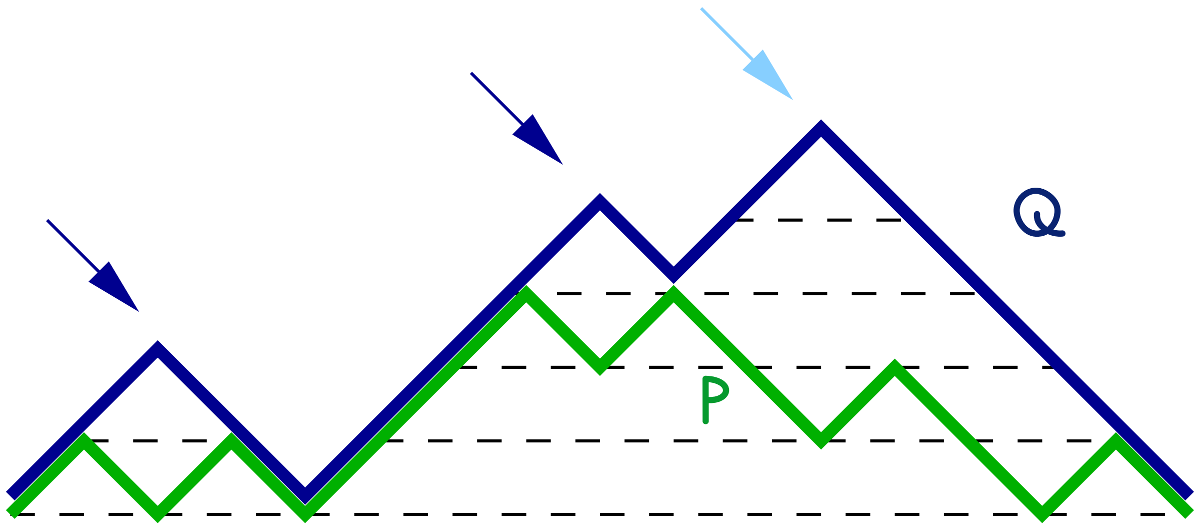


**Corollary:** if [P,Q] is an interval, deleting the last peak of P and the last peak of Q gives a new interval.

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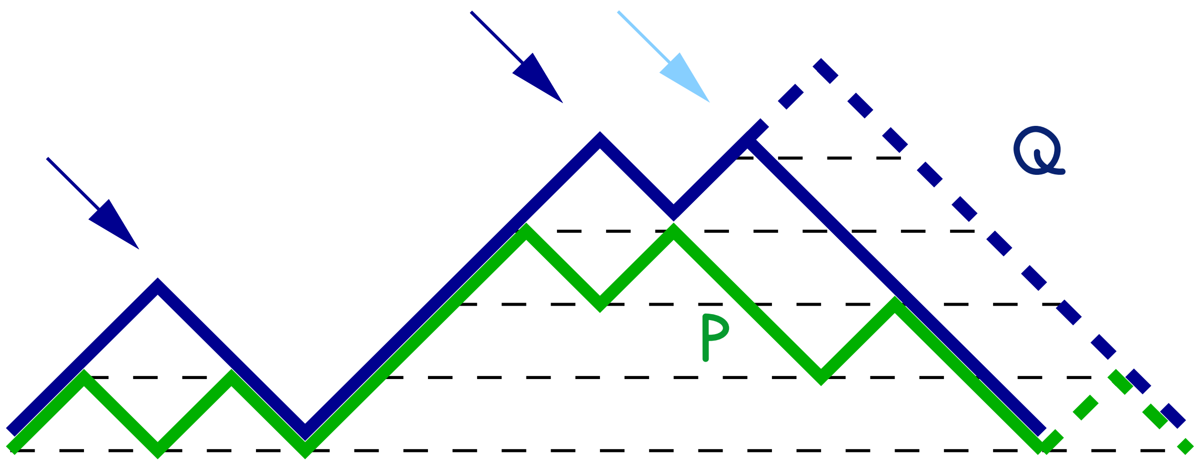


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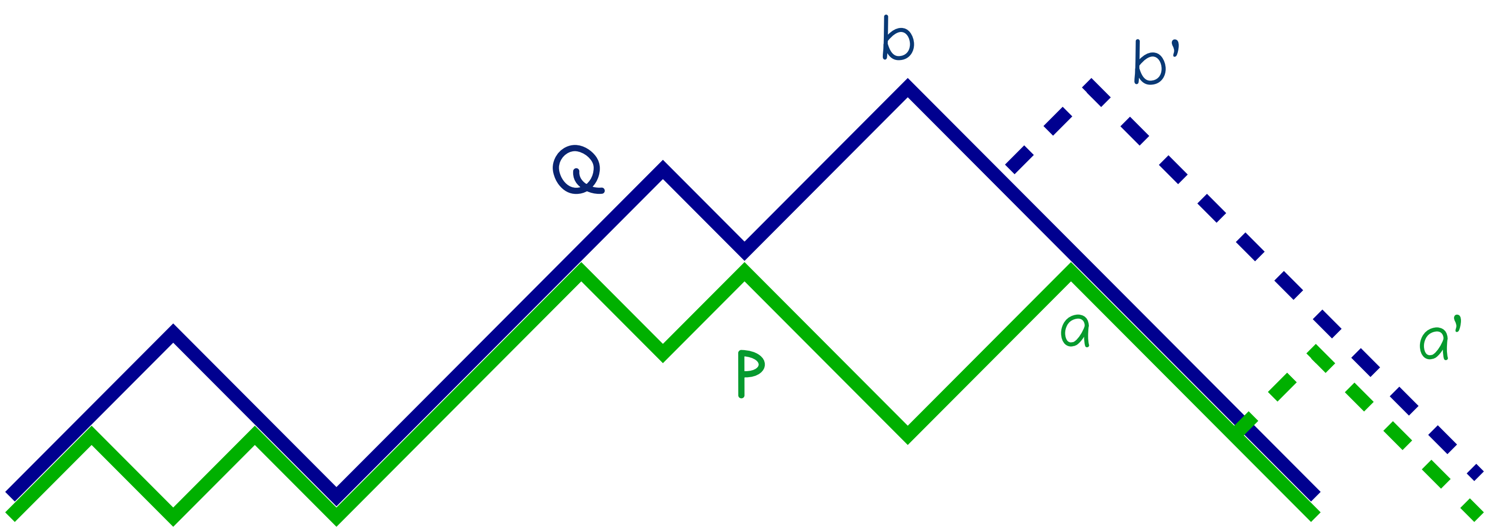
Conversely, starting from an interval [P,Q] with final peaks at heights $a \leq b$, adding peaks in P and Q at heights a' and b' gives an interval iff...



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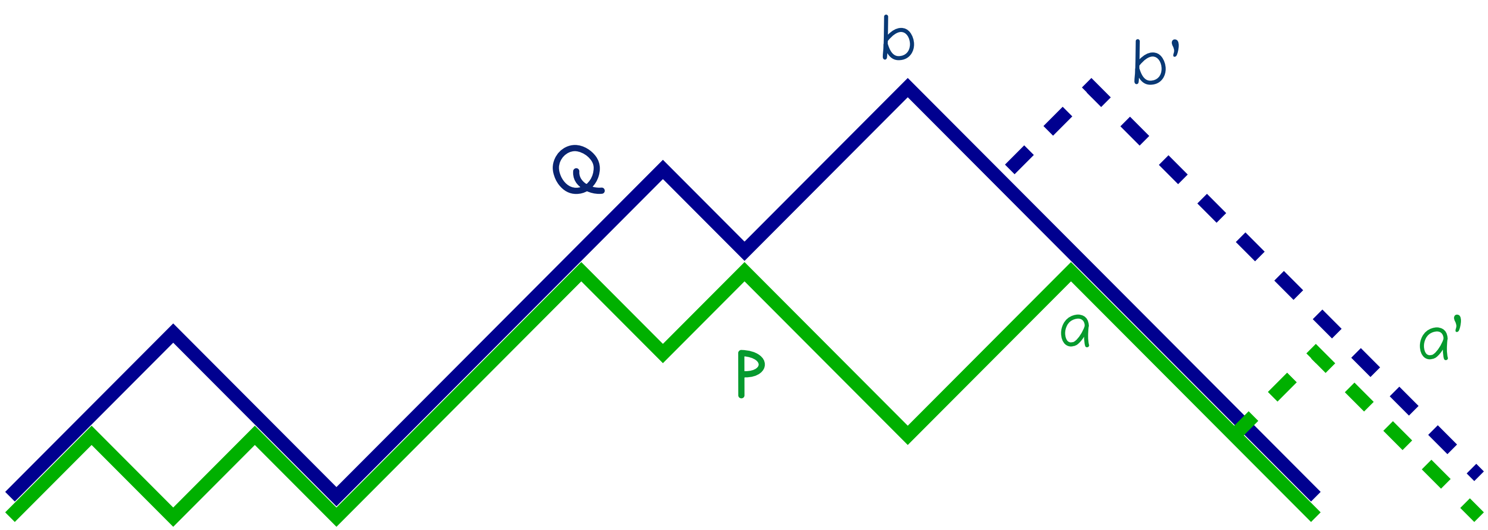

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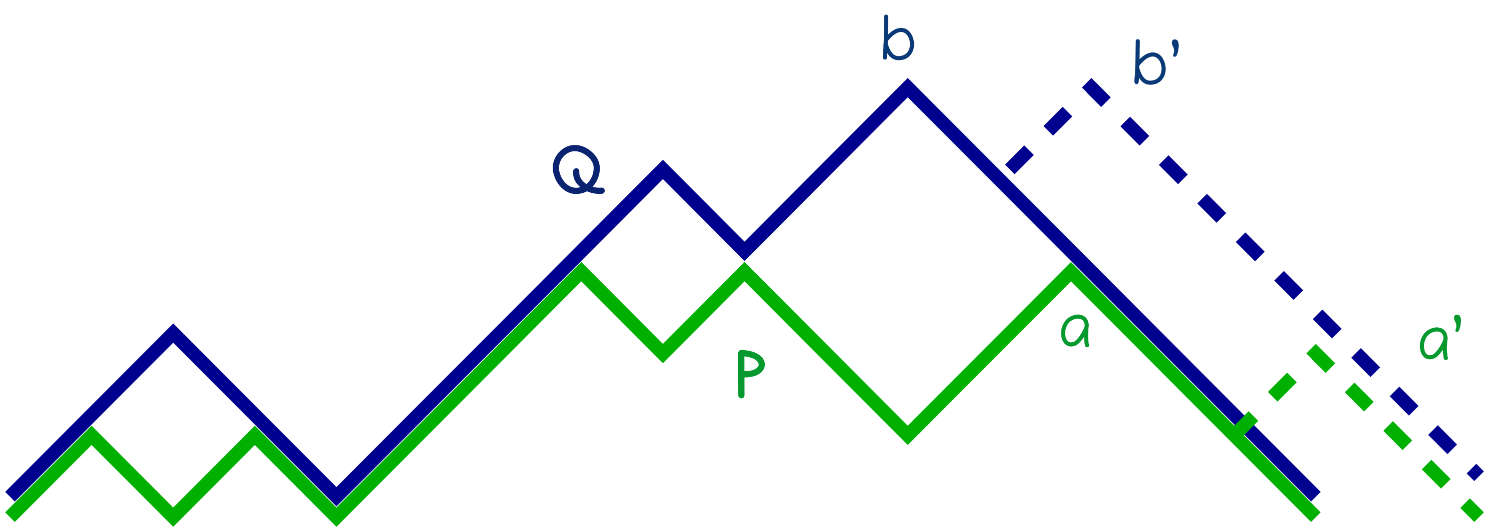

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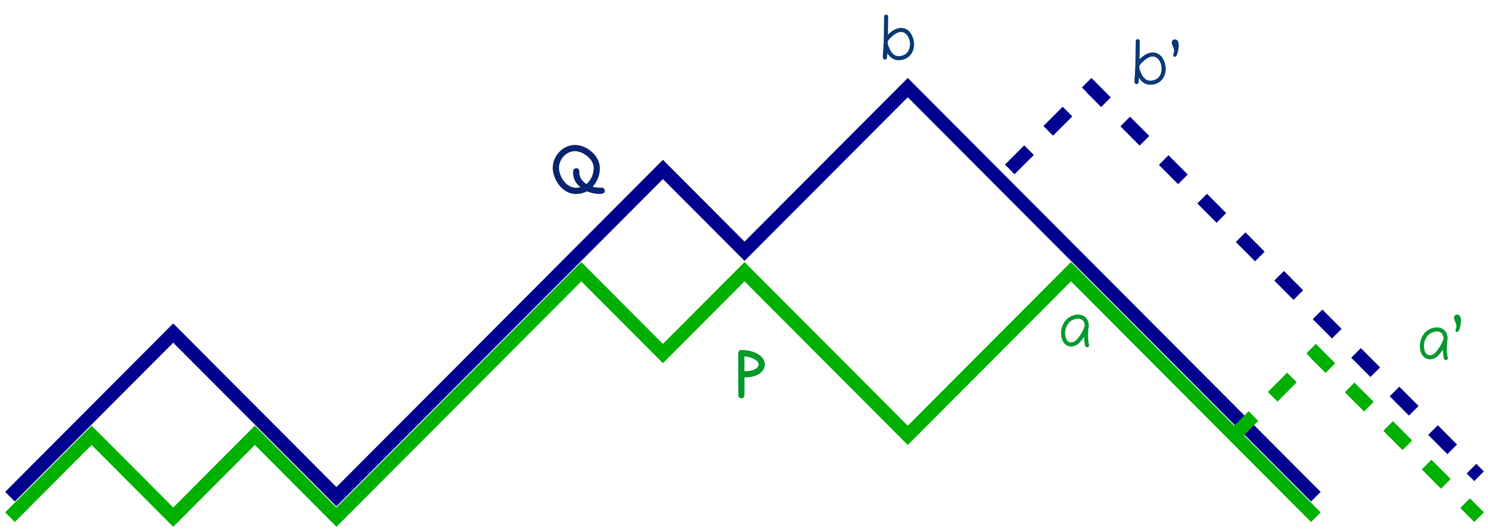

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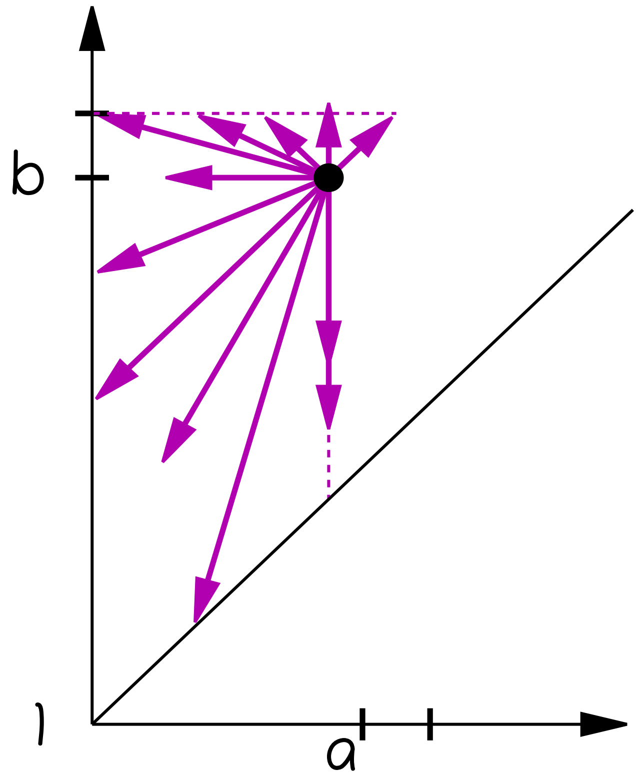

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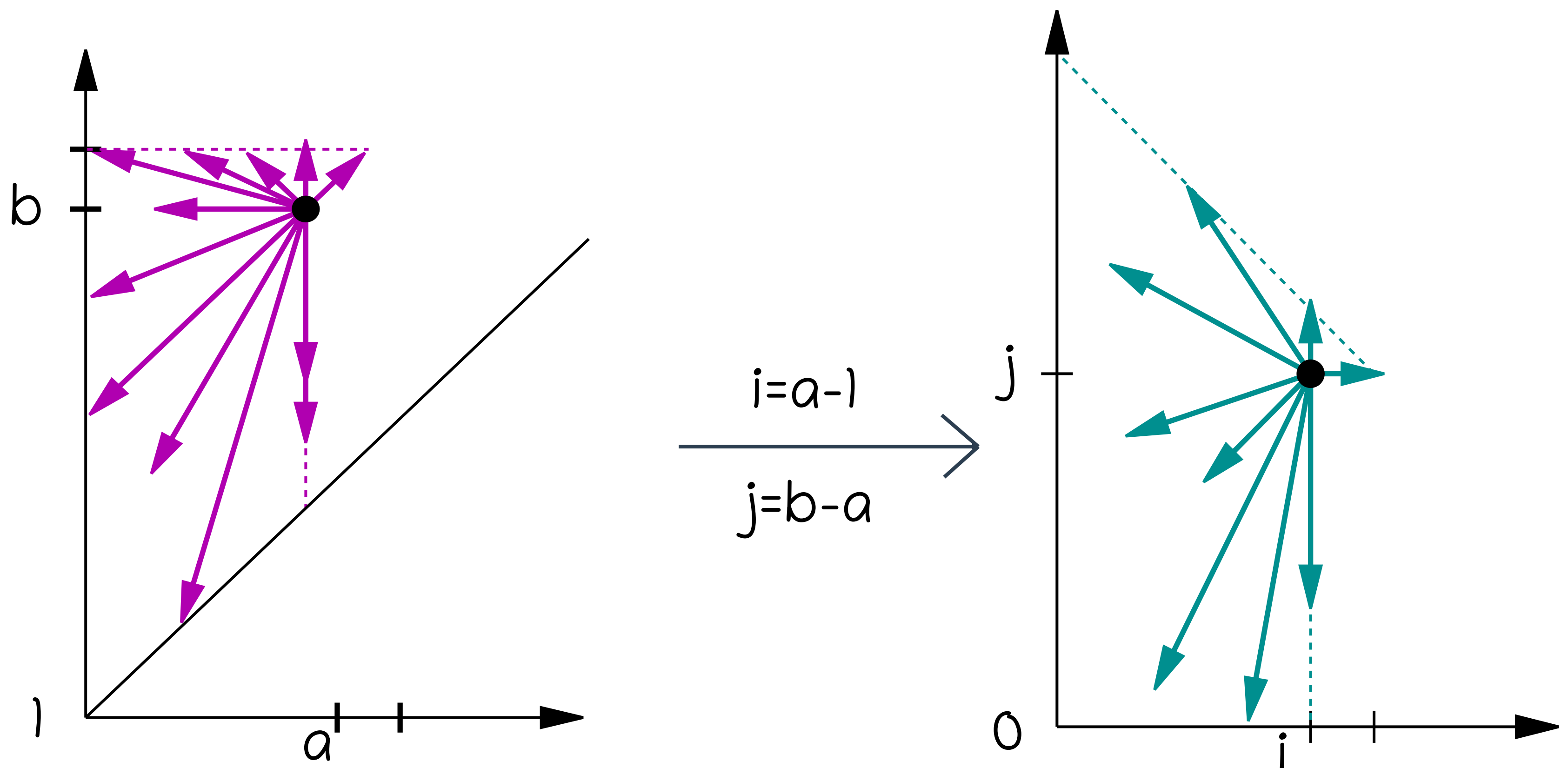

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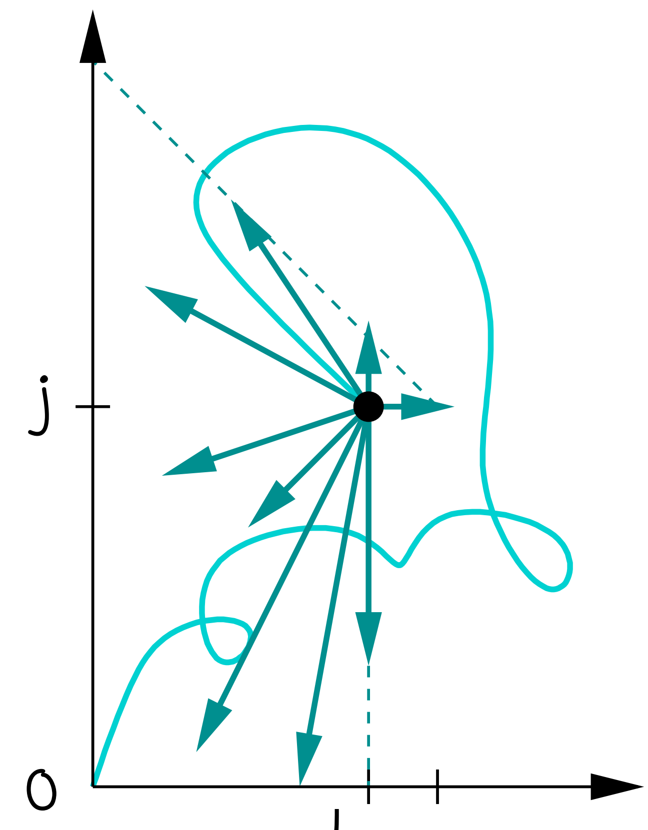
**Bijection** intervals of size n

$\approx$

quadrant walks of length n-1 starting from (0,0)



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- Let Q(t;x,y)=Q(x,y) be the GF of the associated quadrant walks:

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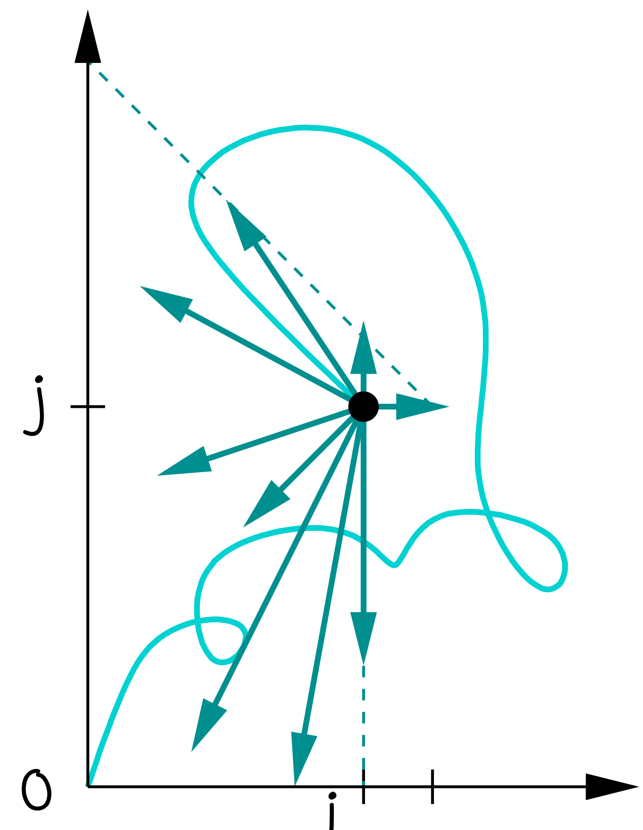

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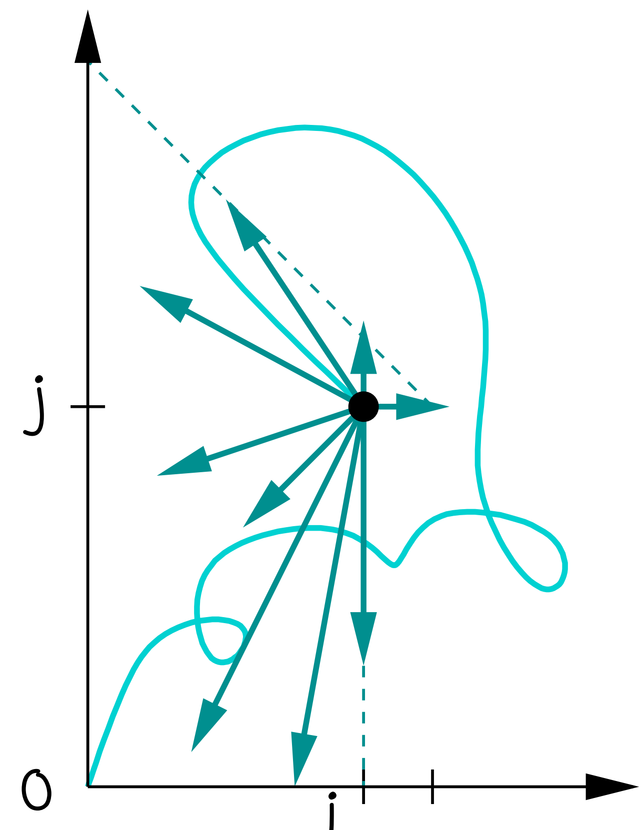

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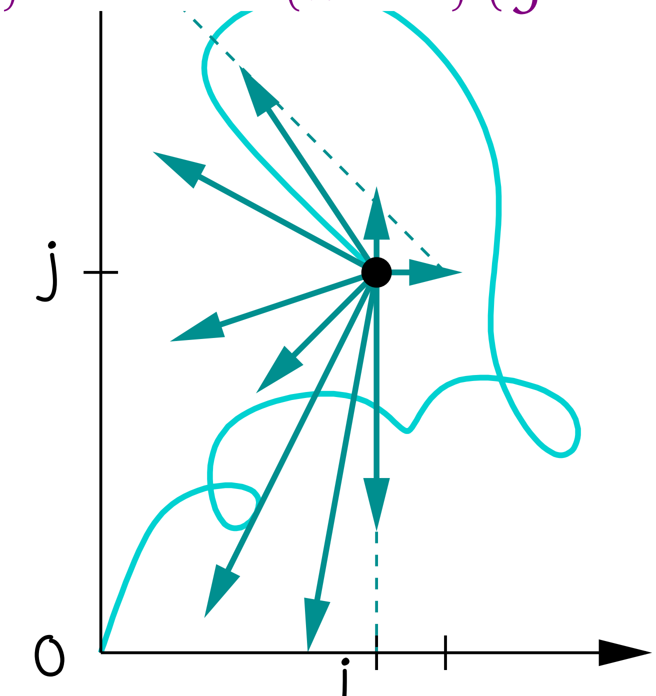

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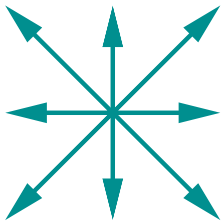
  - **Well understood**: **algebraic/differential** properties of quadrant walks **with finitely many small steps** [Bernardi, Bostan, mbm, Raschel, Mishna, Zeilberger, Kauers, Hardouin, Dreyfus, Roques, Singer, Elvey Price… ]



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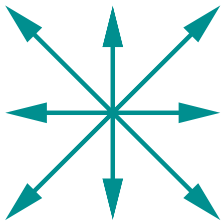

$$\left(1 - tx - ty - \frac{t}{xy}\right) xy\tilde{Q}(x,y) = xy - t\tilde{Q}(0,y) - t\left(\tilde{Q}(x,0) - \tilde{Q}(0,0)\right).$$

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**Thm.** Ascent intervals have **an algebraic GF**, namely

$$G = tQ(1,1) = Z(1 - 2Z + 2Z^3), \quad \text{where} \quad Z = t(1+Z)(1+2Z)^2.$$

**Asymptotics:**

$$g(n) \sim \kappa\, \mu^n n^{-7/2}, \quad \text{with} \quad \mu = \frac{11 + 5\sqrt{5}}{2}.$$

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Tutte's invariants

[Bernardi, mbm, Raschel 21]

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# A functional equation with two "catalytic" variables

- The GF of ascent intervals is tQ(1,1), where Q(x,y)=Q(t;x,y) satisfies:

$$K(x,y)(y-1)Q(x,y) = y - 1 - \frac{ty^3}{x-y} Q(y,y) - t\,\frac{xQ(x,1) - Q(1,1)}{x-1}$$

where

$$K(x,y) = 1 - tx - \frac{txy^2}{(x-y)(y-1)}.$$

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- **Observation**: an equation of the form

$$K(x,y)H(x,y) = I(x) - J(y)$$

would probably be easier to solve.
The pair $(I(x),J(y))$ is **a pair of invariants**.

# (1) Constructing invariants from the equation

$$K(x,y)(y-1)Q(x,y) = y - 1 - \frac{ty^3}{x-y}Q(y,y) - t\,\frac{xQ(x,1)-Q(1,1)}{x-1}$$

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Let

$$I_1(x) = \frac{2+x}{t} + \frac{1}{t^2x} + \frac{1}{t(tx-1)} - \frac{tx}{1-tx}\,\frac{xQ(x,1)-Q(1,1)}{x-1},$$

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This is **a pair of invariants**:

$$I_1(x) - J_1(y) = \frac{(x-y)(y-1)K(x,y)}{1-tx}\left(\frac{xQ(x,y)-yQ(y,y)}{x-y} - \frac{1-txy}{t^2xy(y-1)}\right).$$

# (2) Constructing invariants from the kernel

The kernel:

$$K(x,y) = 1 - tx - \frac{txy^2}{(x-y)(y-1)}.$$

Let

$$I_0(x) = \frac{1}{1-tx} - \frac{1}{tx^2} + \frac{1+t}{tx} + x(1-t) - tx^2$$

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$$I_0(x) - J_0(y) = \frac{(x-y)(1-y+txy)(x+y-xy-xyt(1+x-xy))}{x^2y^2t(xt-1)(y-1)} K(x,y).$$

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This is **a pair of invariants**:

$$I_0(x) - J_0(y) = \frac{(x-y)(1-y+txy)(x+y-xy-xyt(1+x-xy))}{x^2y^2t(xt-1)(y-1)} K(x,y).$$

**Construction?**

A **group of order 10** generated by two birational involutions of (x,y) leaves the kernel unchanged.

Play with the group and the roots of the kernel.

# (3) Relating invariants

Let

$$J_0(y) = -\frac{1}{ty^2} + \frac{1+t}{ty} - \frac{t}{(y-1)^2} + \frac{1-t}{y-1} + y$$

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$$2 - 4t - 2t^2 Q(1,1)$$

(value at y=1).

**Argument:** invariants with no poles are constant

# (4) An equation for Q(y,y) -- Algebraicity

$$J_0(y) + t^3 J_1(y)^2 - t(1+3t) J_1(y) = 2 - 4t - 2t^2 Q(1,1)$$

$$\hookrightarrow \; y^2 t^2 (y-1)^2 Q(y,y)^2 + \left(y\left(2y^2 - 5y + 1\right) t - (y-1)(y-2)\right) Q(y,y) + 2t Q(1,1) + (y-1)(y-2) = 0.$$

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- A single "catalytic" variable, y
- Unknown series Q(y,y) and Q(1,1)
- Systematic algebraic solution [Brown 65, mbm-Jehanne 06]

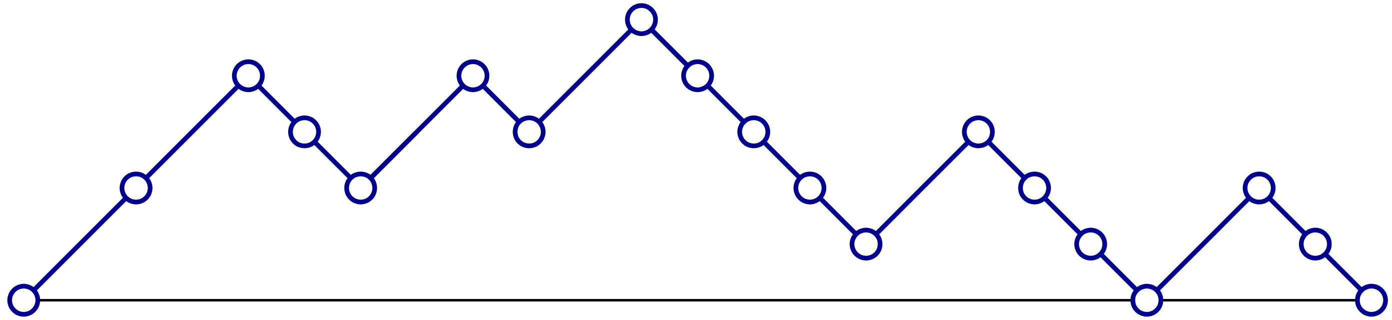
$$64\, t^6 Q_{11}^3 + 16t^3\left(11t^2-18t-1\right) Q_{11}^2 + \left(161t^4 - 452t^3 + 238t^2 - 28t + 1\right) Q_{11} + 49\, t^3 - 167t^2 + 25t = 1.$$

# III. m-Dyck paths, and mirrored m-Dyck paths

# m-Dyck paths and mirrored m-Dyck paths

In an m-Dyck path, the length of each ascent is a multiple of m.

m=2



# m-Dyck paths and mirrored m-Dyck paths

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In a **mirrored m-Dyck path**, the length of each **descent** is a multiple of m.

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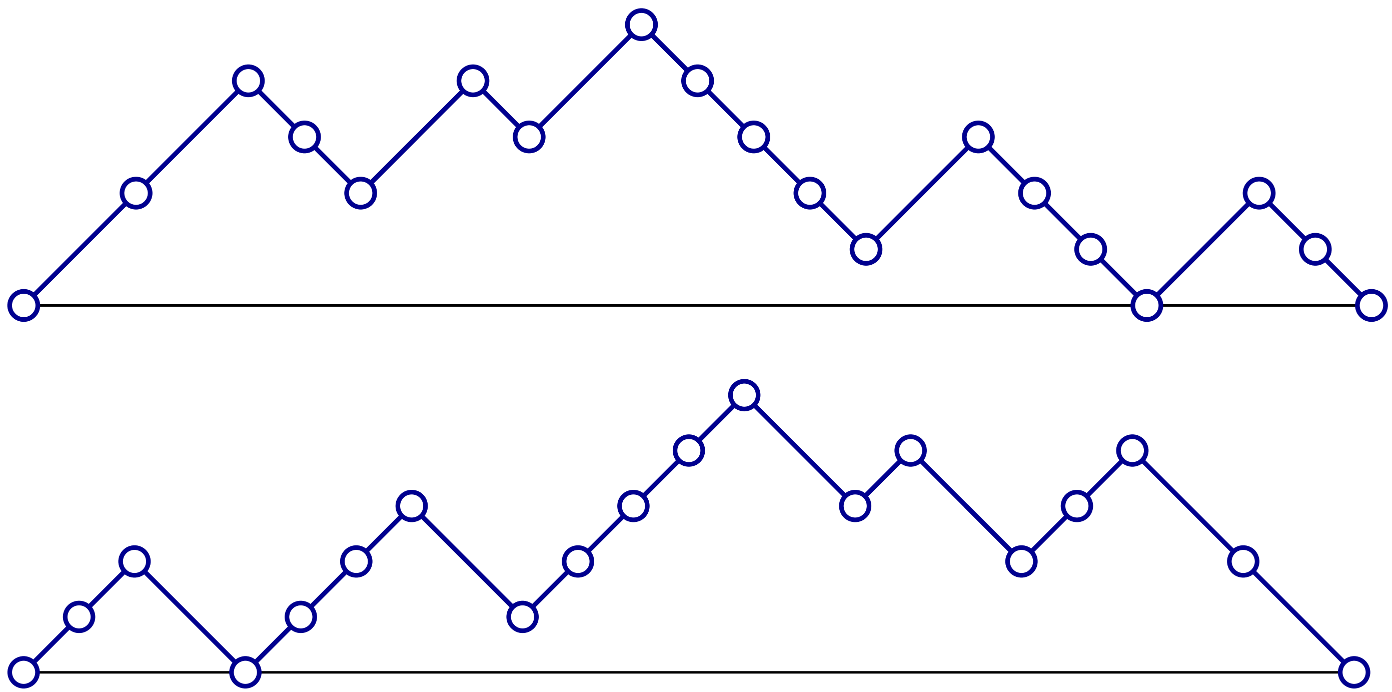

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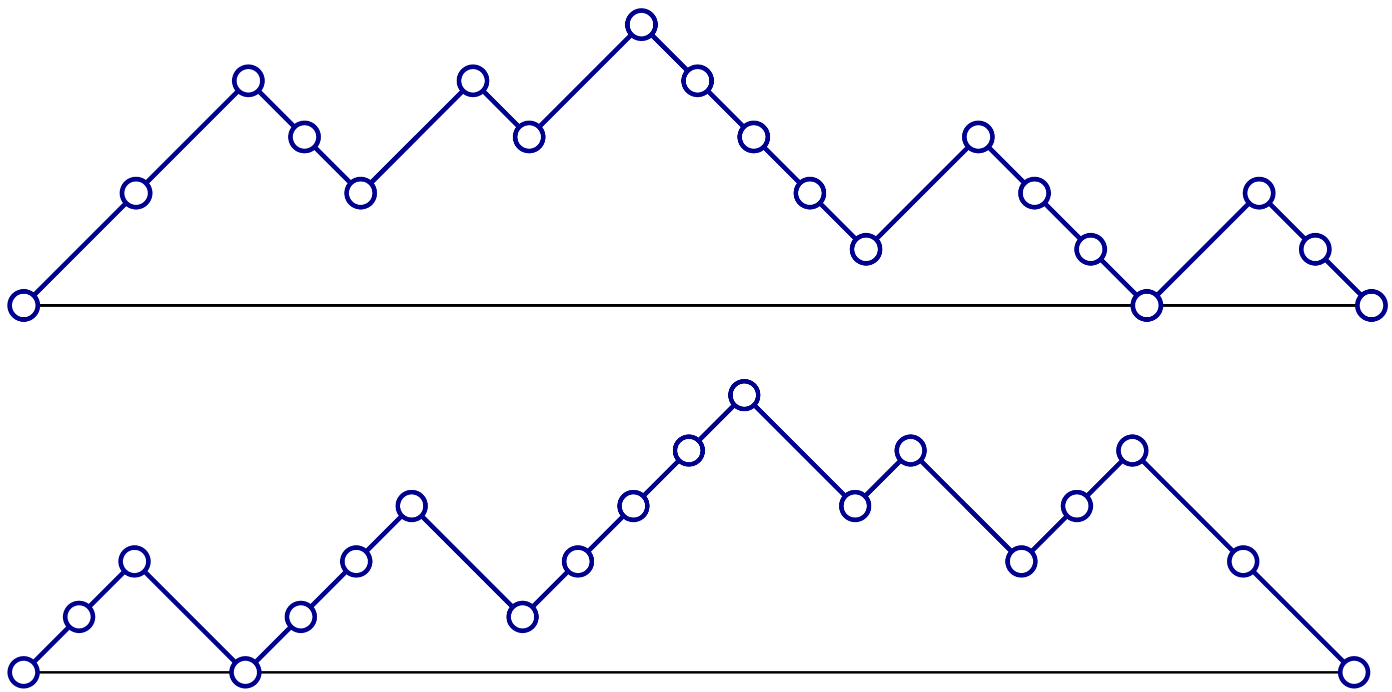
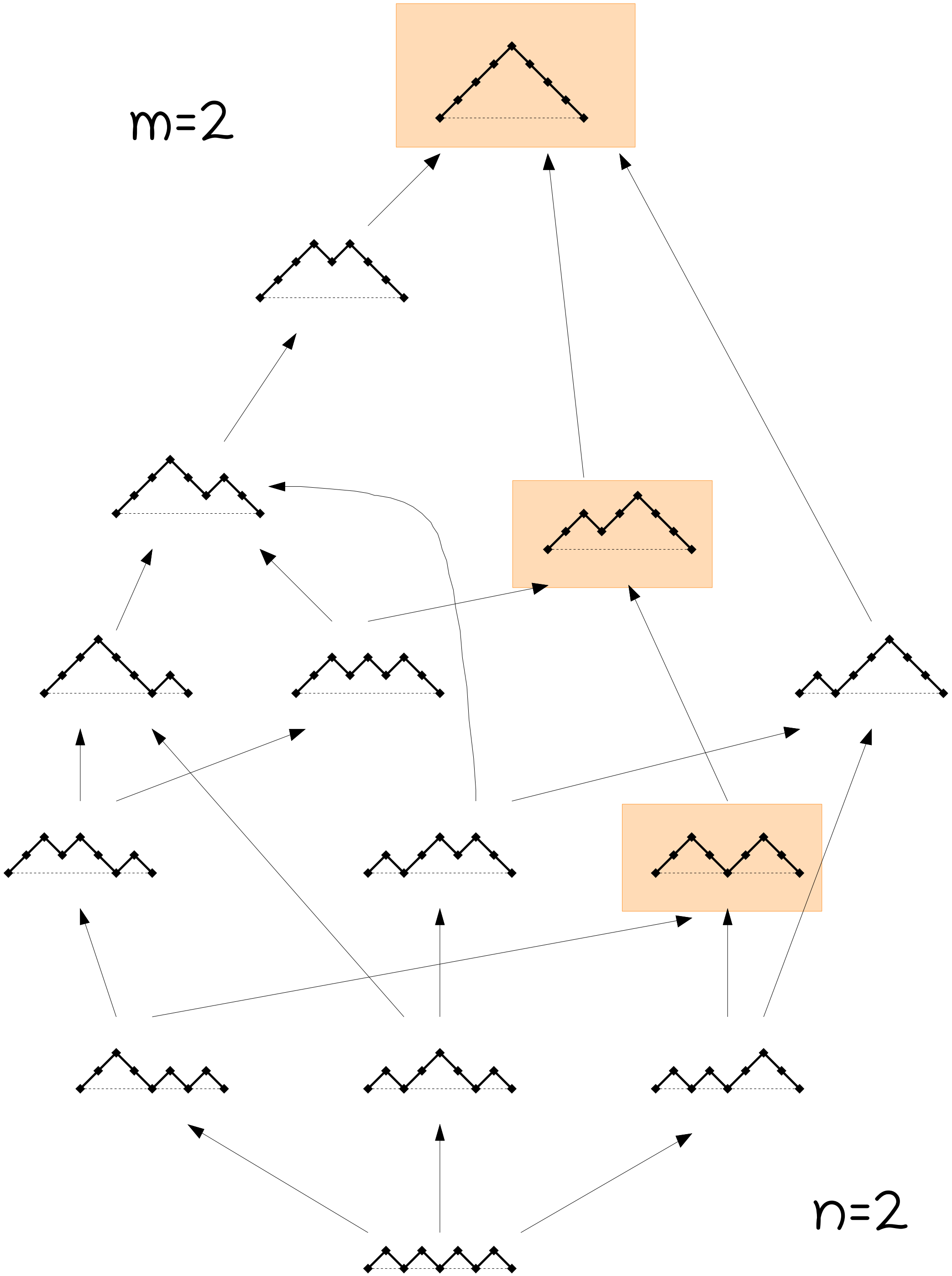
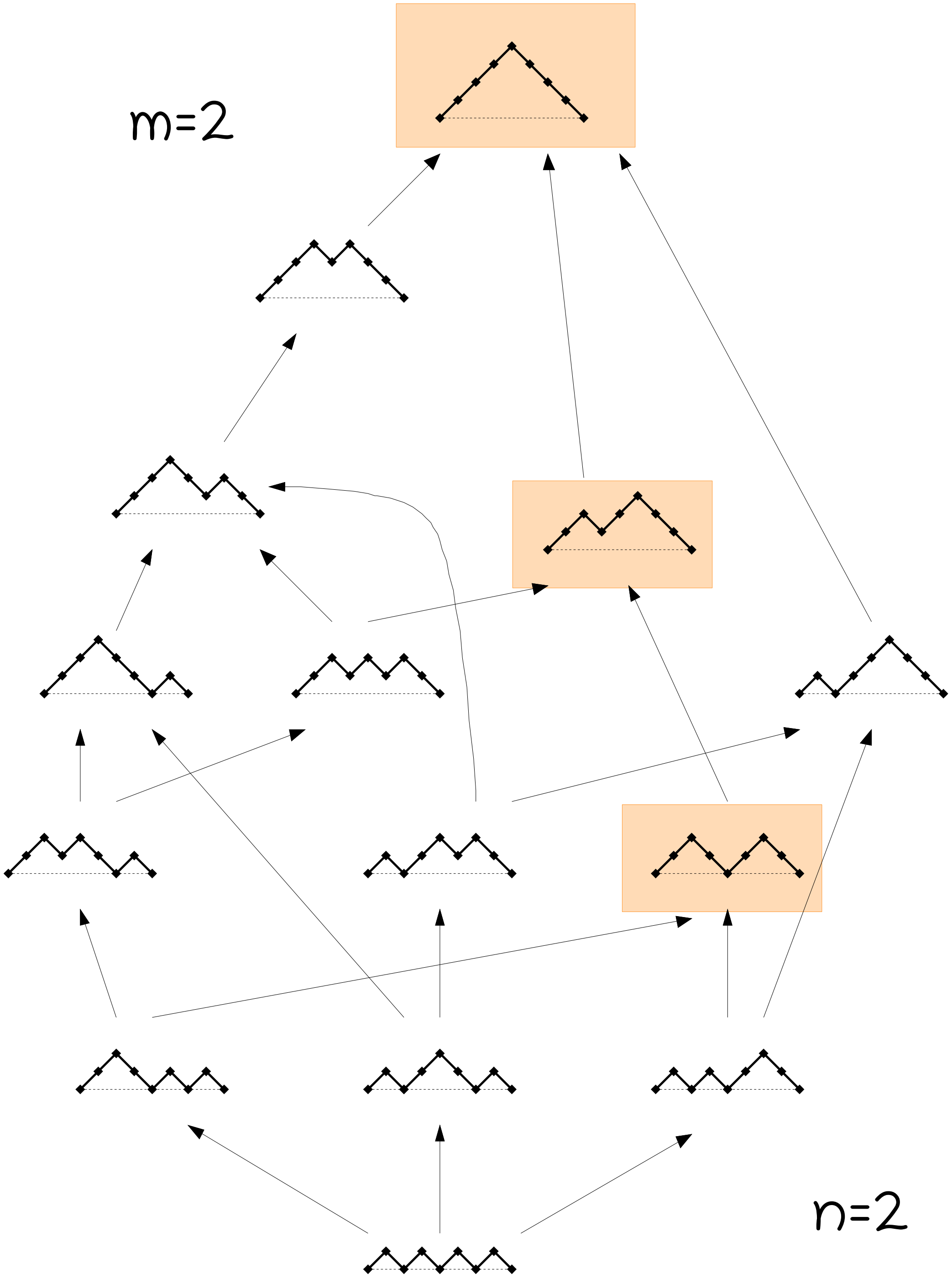


→ Study the order induced by the ascent order on m-Dyck paths and mirrored m-Dyck paths of size mn.

# m-Dyck paths



# m-Dyck paths

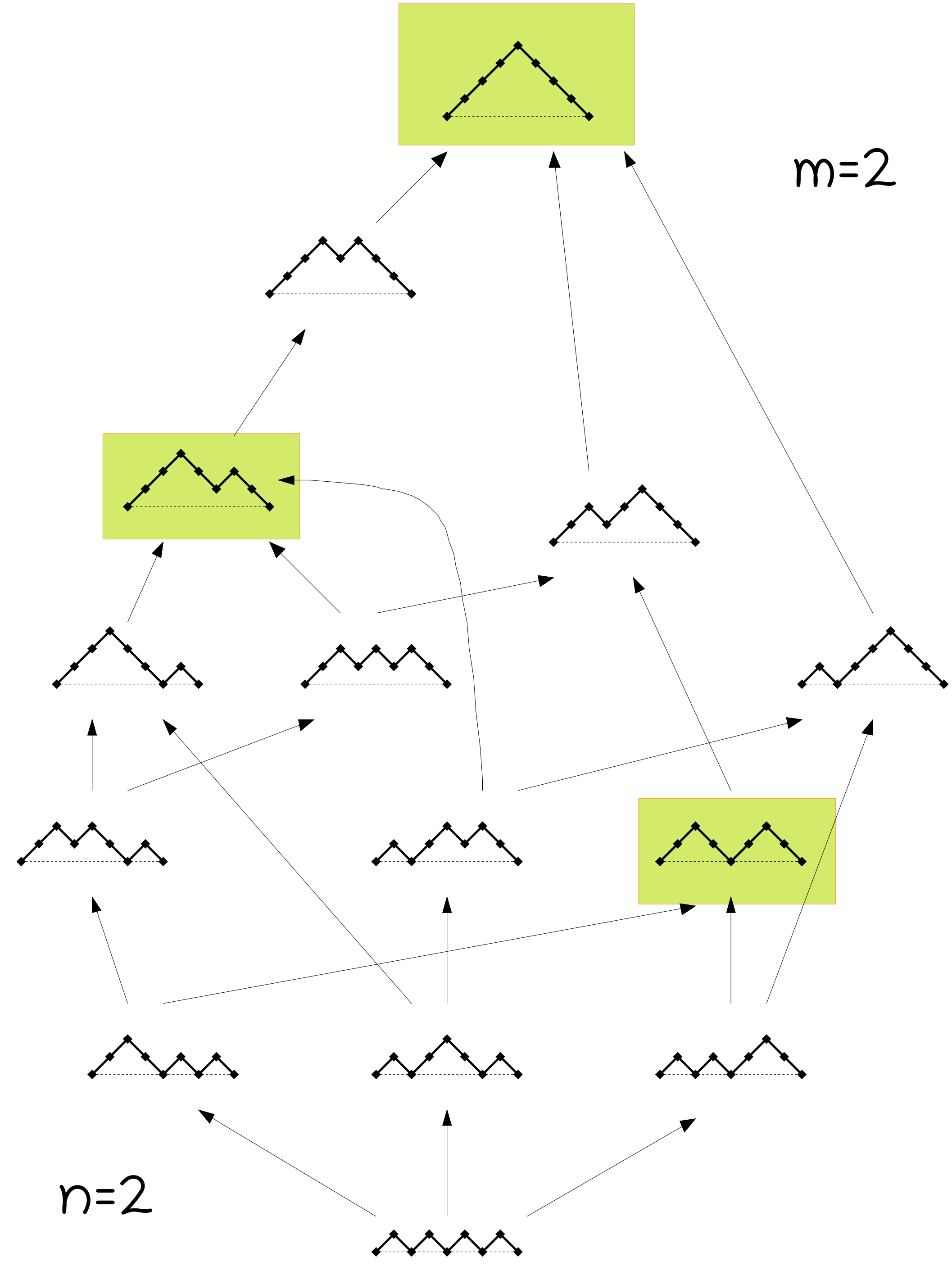



m-Dyck paths form an interval in the ascent lattice $A_{mn}$.

In particular, it is a lattice.

# Mirrored m-Dyck paths

Mirrored m-Dyck paths only form a join semi-lattice.



# What about intervals?

# What about intervals?

## Intervals in m-Dyck paths:

- Stanley lattice: D-finite GF (i.e., linear DE with pol. coeffs)

$$\frac{2(m+2)((m+1)n)!((m+1)(n+1))!}{n!(n+1)!(mn+2)!(m(n+2)+2)!}$$

- Tamari lattice: algebraic GF [mbm, Fusy, Préville-Ratelle 11]

$$\frac{m+1}{n(mn+1)}\binom{(m+1)^2 n+m}{n-1}$$

Conj: Bergeron, Préville-Ratelle

- Greedy Tamari lattice: algebraic GF [mbm, Chapoton 24]

$$\frac{(m+2)(m+1)^{n-1}}{(mn+1)(mn+2)}\binom{(m+1)n}{n}.$$

# Intervals of m-Dyck paths and mirrored m-Dyck paths

## Two families of functional equations

➔ m-Dyck paths: last peak decomposition

$$Q(x,y) = 1 + tx^m Q(x,y) + ty^2 \frac{x^m Q(x,y) - y^m Q(y,y)}{(x-y)(y-1)} - t\frac{x^m Q(x,1) - Q(1,1)}{(x-1)(y-1)}$$

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→ **Mirrored m-Dyck paths**: first peak decomposition

$$\overline{Q}(x,y) = 1 + tx^m \frac{y\overline{Q}(x,y) - \overline{Q}(x,1)}{y-1} + ty^2 \frac{x^m \overline{Q}(x,y) - \overline{Q}(1,y)}{(x-1)(y-1)} - t\,\frac{x^m \overline{Q}(x,1) - \overline{Q}(1,1)}{(x-1)(y-1)}$$

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→ **No exact solution, but explicit asymptotic results** ⇒ not algebraic, not D-finite for m>1 (i.e. no linear diff. equation)

# Intervals in the m-Dyck ascent lattice

- **Asymptotics** (from random walk results) [Denisov & Wachtel 15]

$$g_m(n) \sim \kappa \mu^n n^\alpha,$$

where

$$\mu = \frac{m\sqrt{m^2+4}+m^2+2}{2} \cdot \left(\frac{2+\sqrt{m^2+4}}{m}\right)^m$$

and

$$\alpha = -1 - \pi/\arccos(c) \qquad \text{with} \qquad c = \sqrt{\frac{m^2+2-\sqrt{m^2+4}}{2m^2+6}}.$$

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If the associated GF is D-finite, then $\alpha$ is rational.
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For m>1, the exponent $\alpha$ is irrational, and hence the GF of intervals cannot be D-finite.

Contrast with m-Tamari lattices, where intervals have an algebraic GF

# IV. Connection with the sylvester congruence

[Hivert, Novelli, Thibon 05]

# Intervals of m-Dyck paths and mirrored m-Dyck paths

## Two families of functional equations

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$$Q(x,y) = 1 + tx^m Q(x,y) + ty^2 \frac{x^m Q(x,y) - y^m Q(y,y)}{(x-y)(y-1)} - t\frac{x^m Q(x,1) - Q(1,1)}{(x-1)(y-1)}$$

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→ produce numbers

# In the OEIS...

**Observation**: for $m=1, 2, \ldots, 5$, the sequence $\overline{g}_m(n)$ that counts intervals of mirrored $m$-Dyck paths appears in the OEIS.

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- **m=1:** number of sylvester classes of 1-multiparking functions

Search: **seq:1,3,13,69,417,2759 id:243688**

Displaying 1-1 of 1 result found.

Sort: relevance | references | number | modified | created      Format: long | short | data

A243688      Number of Sylvester classes of 1-multiparking functions of length n.

**1, 3, 13, 69, 417, 2759**

(list; graph; refs; listen; history; text; internal format)

| | |
|---|---|
| OFFSET | 1,2 |
| COMMENTS | See Novelli-Thibon (2014) for precise definition. |
| LINKS | Table of n, a(n) for n=1..6.<br>J.-C. Novelli, J.-Y. Thibon, Hopf Algebras of m-permutations, (m+1)-ary trees, and m-parking functions, arXiv preprint arXiv:1403.5962, 2014. See Fig. 26. |
| KEYWORD | nonn,more |
| AUTHOR | N. J. A. Sloane, Jun 14 2014 |
| STATUS | approved |

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**Observation:** for m=1, 2, ..., 5, the sequence $\overline{g}_m(n)$ that counts intervals of mirrored m-Dyck paths appears in the OEIS.

- **m=1:** number of sylvester classes of 1-multiparking functions
- **m=2:** number of sylvester classes of 2-multiparking functions

Search: **seq:1,5,40,407,4797 id:243671**

Displaying 1-1 of 1 result found.

Sort: relevance | references | number | modified | created      Format: long | short | data

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**1, 5, 40, 407, 4797**

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and so on.

Search: **seq:1,5,40,407,4797 id:243671**

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# The sylvester congruence

- Defined on words on the alphabet $\mathbb{Z}$
- Generated by commutation relations:

$$\mathrm{ac} \cdots \mathrm{b} \equiv \mathrm{ca} \cdots \mathrm{b}, \quad \mathrm{a} \leq \mathrm{b} < \mathrm{c}.$$

- Class representatives: words avoiding subwords acb with $a \leq b < c$,

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**Example:**

2 4 5 3 3 → 4 2 5 3 3

4 2 5 3 3 → **4** 5 2 3 **3**, sylvester word

A general correspondance between sylvester words and intervals of **a larger poset**.

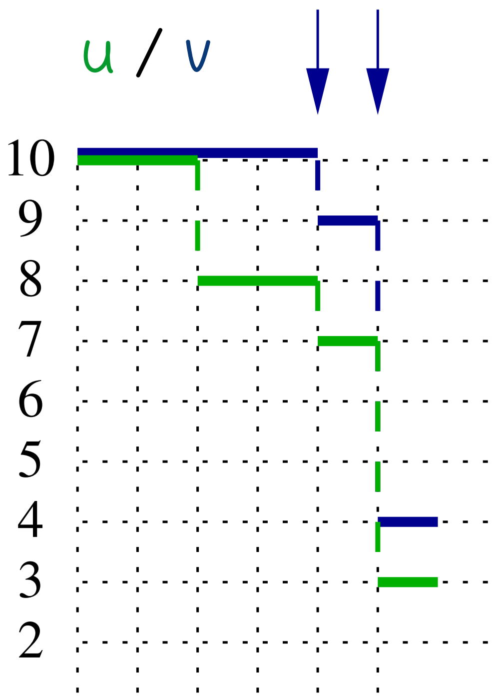
# The Nadeau-Tewari lattice $NT_n$ [2024]

**Def.** Let $u=(u_1, \dots, u_n)$ and $v=(v_1, \dots, v_n)$ be two nonincreasing sequences of integers. Then $u \leq v$ for the NT order if

- $u$ lies below $v$ ($u_i \leq v_i$)
- every descent of $v$ is a descent of $u$.

$v = (10, 10, 10, 10, 9, 4)$

$u = (10, 10, 8, 8, 7, 3)$



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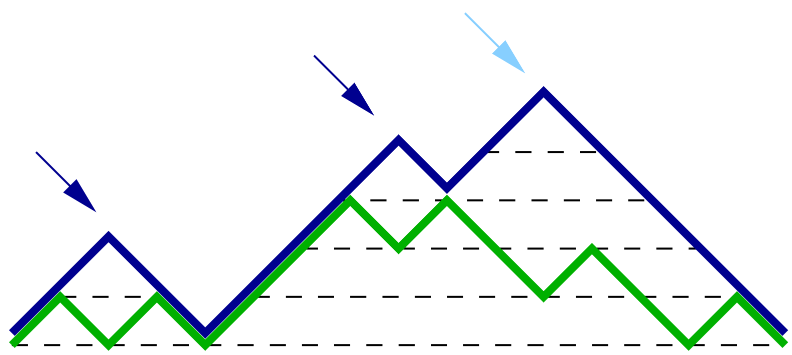

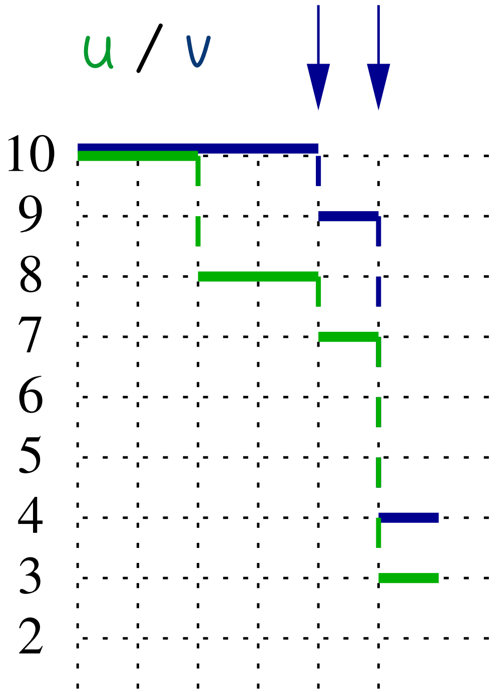

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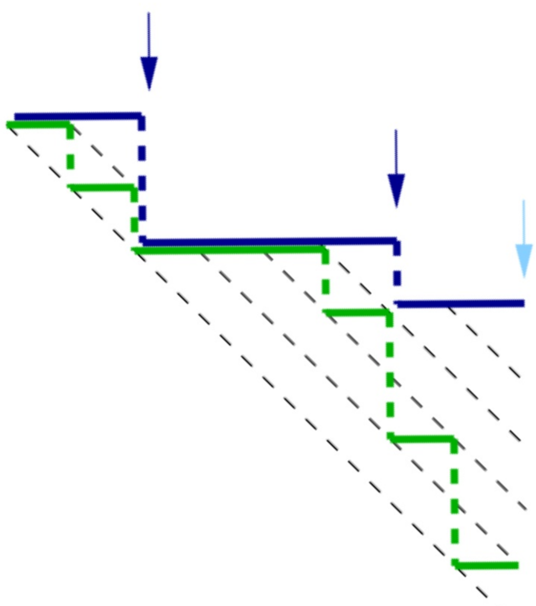

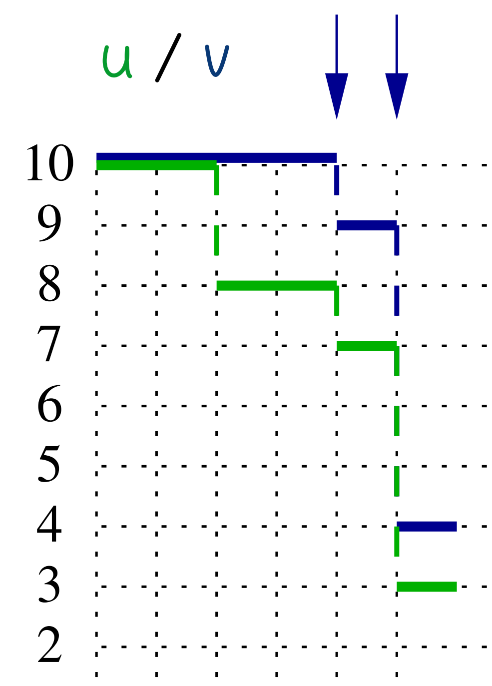

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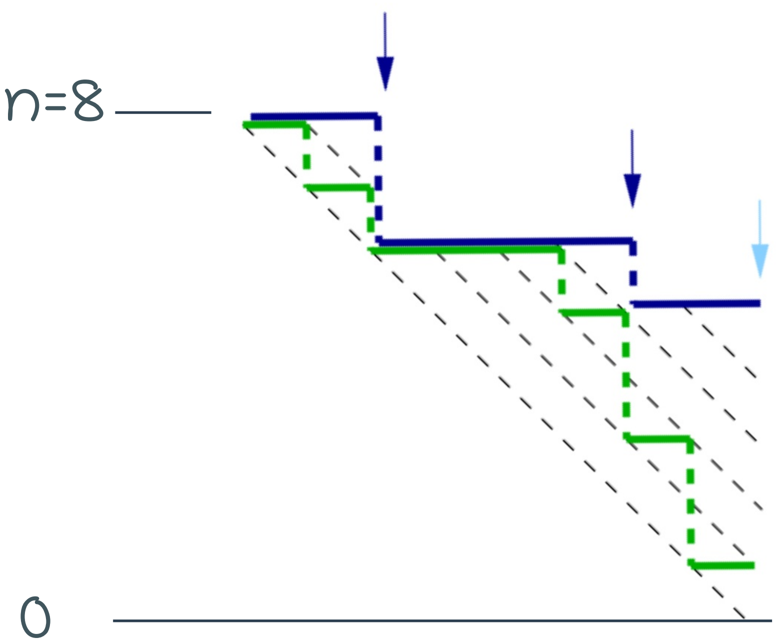
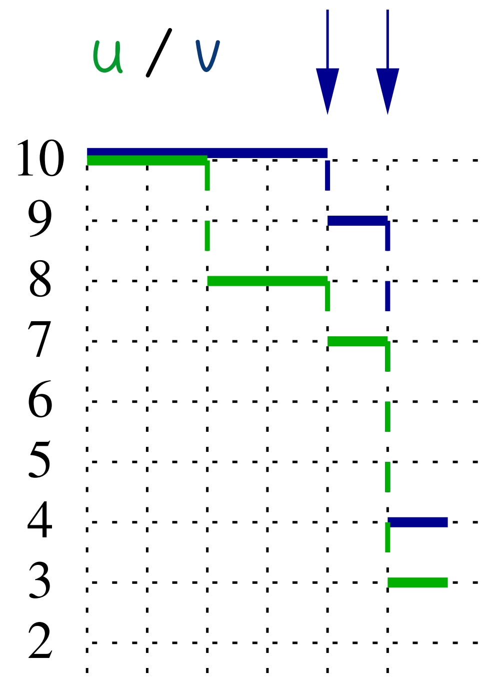



$v = (8, 8, 6, 6, 6, 6, 5, 5)$

$u = (8, 7, 6, 6, 6, 5, 3, 1)$



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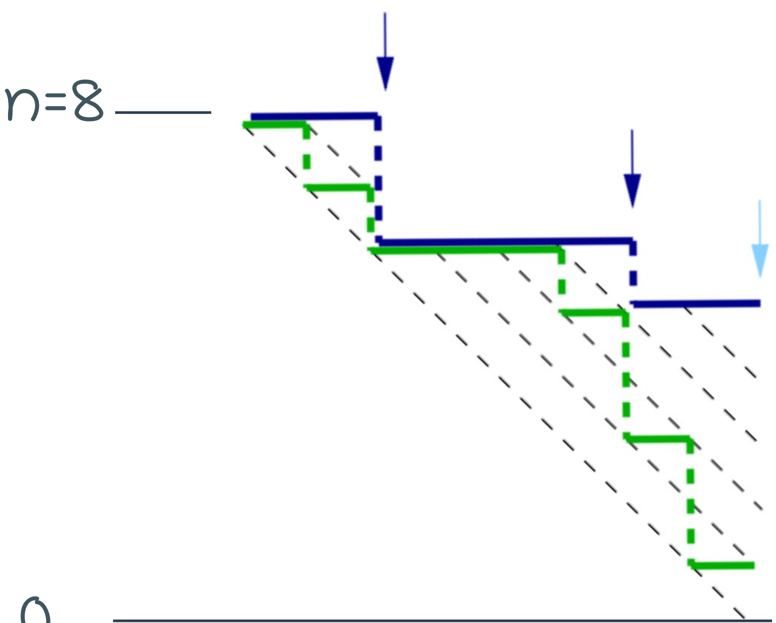
**Observation:** the ascent lattice $A_n$ is the **interval** in the lattice $NT_n$ with $\min=(n,n-1,\dots, 1)$ and $\max=(n,n, \dots, n)$

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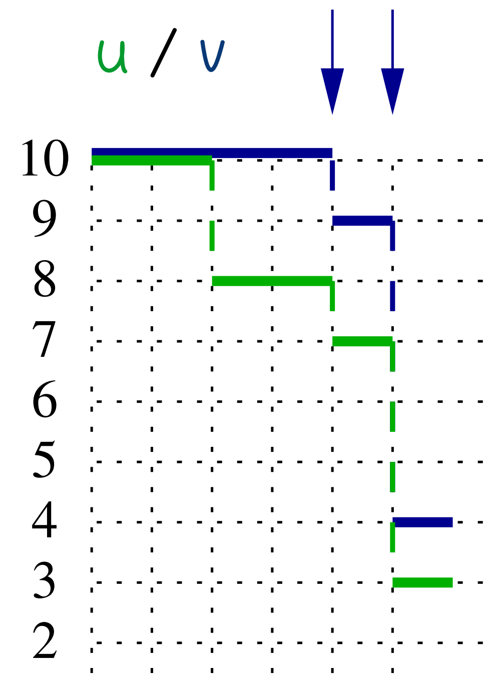

# From sylvester words to Nadeau-Tewari intervals

**Example.** Fix $n=6$ and a sylvester word $w$ on the alphabet $\{1, 2, \ldots, n\}$, containing the letter 1, say $w$ = **3 2** 2 2 2 5 **1** 1 1 5.

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Let $w_1 = \mathrm{NInc}(w) = 5\ 5\ 3\ 2\ 2\ 2\ 2\ 1\ 1\ 1$ be its nonincreasing reordering.

# From sylvester words to Nadeau-Tewari intervals

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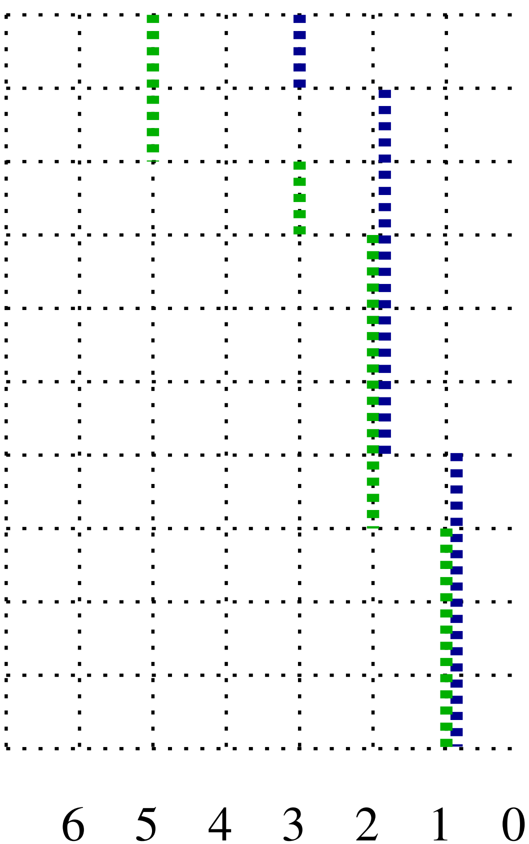

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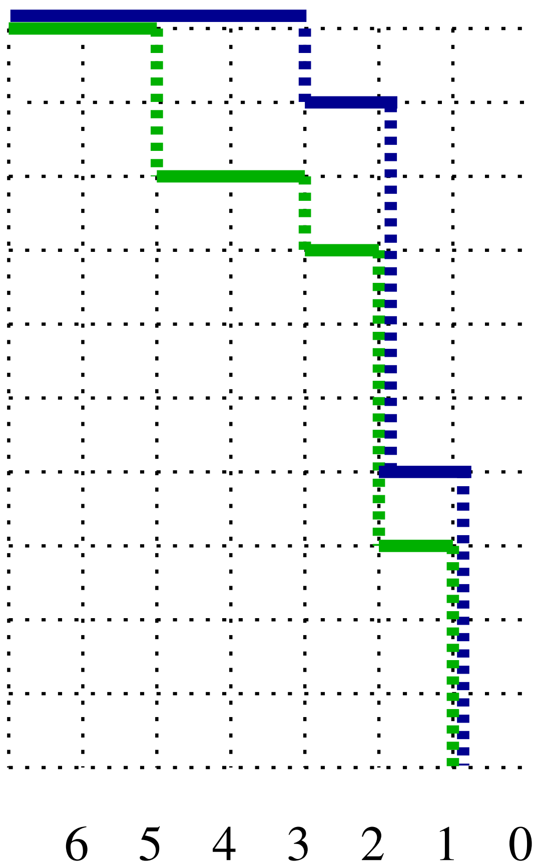

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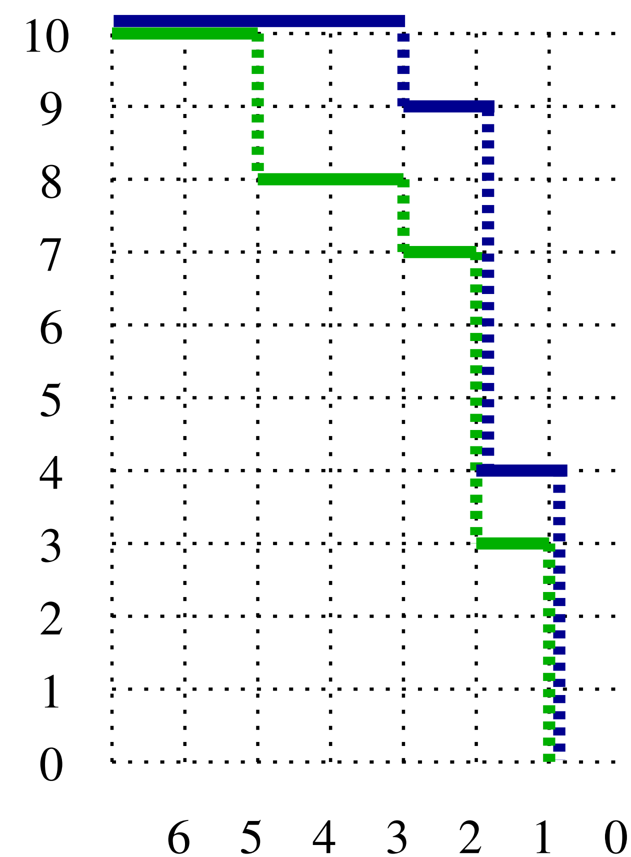

# From sylvester words to Nadeau-Tewari intervals

**Proposition.** For any n, this is a bijection between:

- sylvester words w on the alphabet {1, 2, ..., n} containing the letter 1, and
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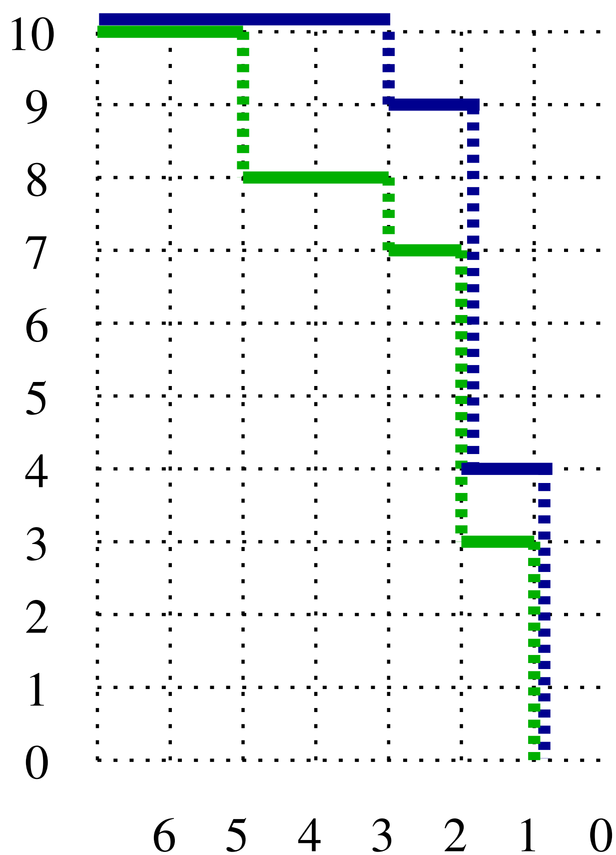
**Example**

For n=6 and w = 3 2 2 2 2 5 1 1 1 5, we have u= 10 10 8 8 7 3 and v= 10 10 10 10 9 4.

Conversely?



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**Specializations:** bijections between

- positive sylvester words $w$ of length $mn$ such that $\mathrm{NInc}(w) \leq n^m\,(n-1)^m \ldots 2^m\,1^m$ and **ascent intervals of m-Dyck paths** of length $mn$
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OEIS

Sylvester classes of m-multiparking functions [Novelli, Thibon 20]

# Final questions

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$$(n+4)(2n+7)\,g(n+2) = 2\left(11n^2+44n+42\right) g(n+1) + n(2n+1)\,g(n)$$

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  a(P) = length of the first ascent of P

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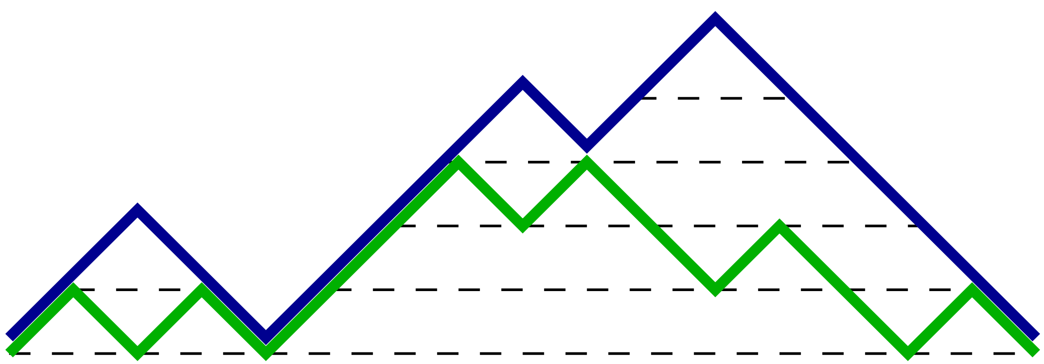
**Example.** a(P)=1, r(P,Q)=2

(Non-recursive) bijection?



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**Thanks for your attention**