Character bounds, card shuffling and random maps

with S. Olesker-Taylor (Warwick) and L. Teyssier (UBC)

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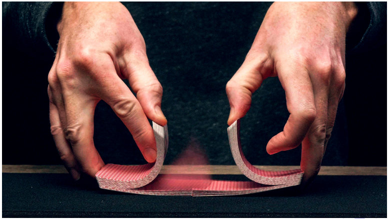

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- What does it mean to be mixed?

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- Card = integer in $[\![1, n]\!]$;
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We consider a random process $(X_k)_{k\geq 0}$ with values in $\mathfrak{S}_n$, such that:

- $X_0 = Id$ (i.e. $X_0(i) = i$ for all $i \in [\![1, n]\!]$)
- $X_{k+1} = (i, j) \circ X_k$, where i, j are integers chosen uniformly at random.

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- Object of interest: $\pi_k^{(n)}$, distribution of X_k. Distribution of the product of k independent uniform transpositions.

Total variation distance

We want to measure how mixed the deck is after a given number of steps.

Definition

Let μ, ν be two probability measures on $\mathfrak{S}_n$. The *total variation distance* between μ and ν is defined as

$$d_{TV}(\mu, \nu) = \frac{1}{2} \sum_{\sigma \in \mathfrak{S}_n} |\mu(\sigma) - \nu(\sigma)| \in [0, 1].$$

Example

- $\mu = \delta_{Id_n}$: $\forall \sigma \in \mathfrak{S}_n, \mu(\sigma) = \mathbb{1}_{\sigma = Id_n}$.
- $\nu = Unif_{\mathfrak{S}_n}$: $\forall \sigma \in \mathfrak{S}_n, \nu(\sigma) = \frac{1}{n!}$.

We have

$$d_{TV}(\mu, \nu) = \frac{1}{2} \sum_{\sigma \in \mathfrak{S}_n} |\mu(\sigma) - \nu(\sigma)|$$

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Mixing time and cutoff phenomenon

Let $\pi_k^{(n)}$ be the distribution of X_k, and set $d_n(k) := d_{TV}\left(\pi_k^{(n)}, Unif_{\mathfrak{S}_n}\right)$.

Theorem [Diaconis-Shahshahani '81]

Fix $\varepsilon > 0$.

- $d_n\left((1-\varepsilon)\frac{1}{2}n\ln(n)\right) \to 1$.
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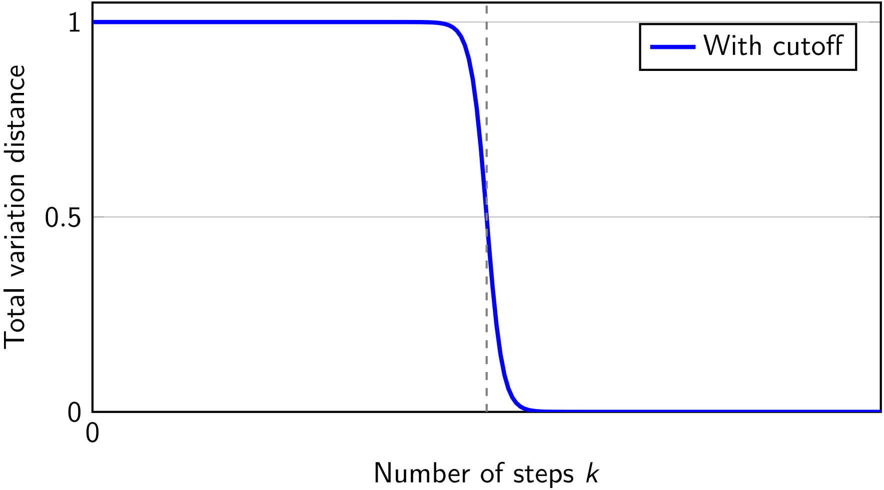
- Proof : representation theory.
- Heuristic: for $k(n) \ll n\ln(n)$, $X_{k(n)}$ has too many fixed points.

Cutoff



1
0.5
0
0
Total variation distance
Number of steps k
With cutoff

Profile

The phase transition occurs at a smaller time scale

Theorem [Teyssier '20]

There exists $f : \mathbb{R} \mapsto [0,1]$ decreasing from 1 to 0 such that, for all $c \in \mathbb{R}$:

$$d_n\left(\frac{1}{2}n\ln(n) + cn\right) \underset{n\to\infty}{\to} f(c).$$

- $f(c) = d_{TV}\left(Po(1+e^{-2c}), Po(1)\right)$.

Extensions

Instead of multiplying transpositions, multiply other permutations ?

Definition

The conjugacy class of $\mathfrak{S}_n$ denoted $\left[a_1^{k_1}, a_2^{k_2} \ldots\right]$ $(a_1 \geq a_2 \geq \ldots)$ is the set of permutations of $\mathfrak{S}_n$ with this cyclic structure : k_1 cycles of length a_1, k_2 cycles of length a_2, etc.

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- $[2, 1^{n-2}]$: transpositions
- $[n]$: n-cycles
- $[2^{n/2}]$ (n even) : product of $n/2$ transpositions.

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We are interested in the law $\pi_k^{(n)}$ of the product of uniform elements of $\mathcal{C}_n$, where $\mathcal{C}_n$ is a conjugacy class of $\mathfrak{S}_n$. Mixing time? Cutoff? Profile?

(Non-extensive) bibliography

- Transpositions
  - Cutoff [Diaconis-Shahshahani '81]
  - Phase transition [Berestycki-Durrett '06]
  - Probability of having long cycles [Alon-Kozma '13]
  - Profile [Teyssier '20], [Jain-Shawney '24]
- Other conjugacy classes
  - Some fixed-point-free conjugacy classes [Lulov '96]
  - Mixing time for fixed-point-free conjugacy classes [Larsen-Shalev '08]
  - Cutoff for k-cycles, k finite [Berestycki-Schramm-Zeitouni '11]
  - Cutoff for k-cycles, $k = o(n)$ [Hough '16]
  - Cutoff for conjugacy classes, support $= o(n)$ [Berestycky-Şengül '19]
  - Profile, support $= o(n)$ [Nestoridi, Olesker-Taylor '21].

Conjugacy classes with many fixed points

Let $\mathcal{C}_n$ be a conjugacy class of $\mathfrak{S}_n$. Denote by:

- f_n the number of fixed points of an element of $\mathcal{C}_n$;
- $t_n := \frac{\ln(n)}{\ln(n/f_n)}$;
- $d_n(k) = d_{TV}\left(\pi_k^{(n)}, Unif_{\mathfrak{S}_n}\right)$.

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Theorem [Olesker-Taylor, Teyssier, T. '25]

Fix $\varepsilon > 0$. Uniformly for any class $\mathcal{C}_n$ with at least $\frac{n}{(\ln(n))^{1/4}}$ fixed points, we have:

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- For $\mathcal{C}_n$ the class of transpositions, we recover $f_n = n-2$, $t_n \sim \frac{1}{2} n \ln(n)$.

A profile result

Theorem [Olesker-Taylor, Teyssier, T. '25]

Fix $\varepsilon > 0$. For any fixed $c \in \mathbb{R}$, uniformly for any class $\mathcal{C}_n$ with at least $\frac{n}{(\ln(n))^{1/4}}$ fixed points, we have:

$$d_n\left(t_n\left(1 + \frac{c}{\ln(n)}\right)\right) \to f(c).$$

- Same profile as in the case of transpositions.

Permutations without fixed points

Consider now conjugacy classes without fixed points.

- $[n]$: uniformly random cycles of length n;
- $[2^{n/2}]$: product of $n/2$ transpositions.

After how many steps are we close to uniform?

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In particular, are we uniform after 2 steps?

The n-cycles

Take two uniform n-cycles σ_1, σ_2. Is the product $\sigma_1 \circ \sigma_2$ almost uniform ?

$$(123456789) \cdot (125489367) = (137268)(49)(5)$$

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$$\boxed{\textit{YES}!}$$

Product of $n/2$ transpositions

Take τ_1, τ_2 two uniform products of $n/2$ transpositions. Is $\tau_1 \circ \tau_2$ almost uniform?

$$(12)(34)(56)(78) \cdot (18)(23)(47)(56) = (173)(248)(5)(6).$$

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For all i, $\mathbb{P}(\tau_1 \circ \tau_2(i) = i) = \frac{1}{n}$. But whenever we create a fixed point, we create a second one at the same time.

$$\boxed{NO!}$$

Mixing time

Theorem [Larsen & Shalev '08]

Fixed-point-free conjugacy classes mix in either 2 or 3 steps.

- 2 for n-cycles ;
- 3 for products of $n/2$ transpositions.

Our results

Theorem [Teyssier & T. '24]

For all $n \geq 1$, let $\mathcal{C}_n$ be a conjugacy class of $\mathfrak{S}_n$ without fixed point. Then, the mixing time for $(\mathcal{C}_n)_{n\geq 1}$ is 2 if and only if the number of transpositions in $\mathcal{C}_n$ is $o(n)$.

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- Larger cycles play no role.
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- Proof: representation theory.

Elements of proof

Lemma [Diaconis & Shashahani '81]

Let $\mathcal{C}$ be a conjugacy class of $\mathfrak{S}_n$. Then, for all $k \geq 1$:

$$4d_n(k)^2 \leq \sum_{\lambda \in \widehat{\mathfrak{S}_n}} \left(d_\lambda \left| \chi^\lambda(\mathcal{C}) \right|^k \right)^2 .$$

- the λ denote the irreducible representations of $\mathfrak{S}_n$;
- $\chi_\lambda(\mathcal{C})$ is the (renormalized) character of the representation λ at $\mathcal{C}$;
- d_λ is the dimension of the representation.

Computing the dimension: the hook-length formula

There are combinatorial formulas to compute d_λ and $\chi^\lambda(\mathcal{C})$.

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Lemma

There is a bijection between irreducible representations of $\mathfrak{S}_n$ and *Young diagrams* of size n.

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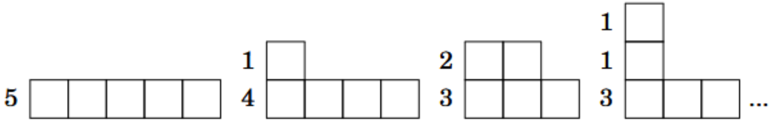
Young diagrams of size n : arrays with n boxes, where each row has less boxes than the one below.

Exemple : $n = 5 = 4 + 1 = 3 + 2 = 3 + 1 + 1 = \ldots$



Computing the dimension d_λ

Lemma

Let λ be an irreducible representation of $\mathfrak{S}_n$. Then we have:

$$d_\lambda = |ST_n(\lambda)|,$$

where $|ST_n(\lambda)|$ is the number of standard Young tableaux of shape λ.

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$$\begin{array}{|c|c|c} \hline 4 & 5 & \multicolumn{1}{c}{} \\ \hline 1 & 2 & \multicolumn{1}{c|}{3} \\ \hline \end{array} \qquad \begin{array}{|c|c|c} \hline 3 & 5 & \multicolumn{1}{c}{} \\ \hline 1 & 2 & \multicolumn{1}{c|}{4} \\ \hline \end{array} \qquad \begin{array}{|c|c|c} \hline 3 & 4 & \multicolumn{1}{c}{} \\ \hline 1 & 2 & \multicolumn{1}{c|}{5} \\ \hline \end{array}$$

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$\lambda = (3, 2)$, $d_\lambda = 5$.

Hooks

Let u be a box of the Young diagram λ. The hook $H(\lambda, u)$ is the set of boxes above or to the right of u (u included).

$$\left|H(\lambda, u)\right| = 5.$$

Hook-length formula

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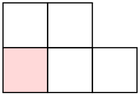
Let λ be an irreducible representation of $\mathfrak{S}_n$. Then we have:

$$d_\lambda = |ST_n(\lambda)| = \frac{n!}{\prod_{u\in\lambda} |H(\lambda, u)|}.$$



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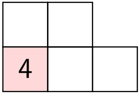

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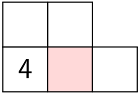

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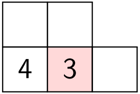

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2	1	
4	3	1

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$$d_\lambda = \frac{5!}{4 \cdot 3 \cdot 1 \cdot 2 \cdot 1} = 5.$$

Character bounds

We now want to bound the characters $|\chi_\lambda(\mathcal{C})|$.

Theorem [Larsen & Shalev '08]

For all $\lambda, \mathcal{C}$, we have

$$d_\lambda |\chi^\lambda(\mathcal{C})| \leq D(\lambda)^{E(\mathcal{C})},$$

where $E(\mathcal{C})$ is easy to compute, and $D(\lambda)$ is the *virtual degree* of λ.

Virtual degree

1					
3	2	1			
6	4	3	1		
9	7	6	4	2	1

1					
2	1	1			
3	1	2	1		
1	5	4	3	2	1

$$d_{\lambda} = \frac{n!}{\prod |H(u,\lambda)|}$$

$$D(\lambda) = \frac{(n-1)!}{\prod a_i! \prod b_i!}$$

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Bounding the virtual degree

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Theorem [Teyssier & T. '24]

As $n \to \infty$, uniformly for all λ of size n, we have

$$D(\lambda) = d_\lambda^{1+O\left(\frac{1}{\ln n}\right)}.$$

Sharpness

This $O\left(\frac{1}{\ln n}\right)$ is sharp. Example: diagrams with 2 boxes above the first row, $\lambda = [n-2, 2]$.

2	1			
7	5	3	2	1

2	1			
1	4	3	2	1

$$d_\lambda = \frac{n!}{2*(n-1)(n-2)(n-4)!)} \qquad D(\lambda) = \frac{(n-1)!}{(n-3)!} \qquad ,$$

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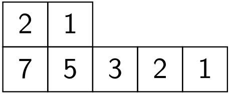

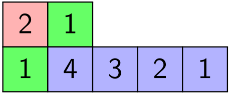

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so $D(\lambda) = d_\lambda^{1+\frac{\ln(2)/2+o(1)}{\ln n}}$.

Idea of the proof

Define a third notion of dimension: the *augmented dimension* d^+_λ.

1					
3	2	1			
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Idea of the proof

Define a third notion of dimension: the *augmented dimension* d_λ^+.

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Lemmas

Theorem [Teyssier & T. '24]

As $n \to \infty$, uniformly for all λ of size n, we have

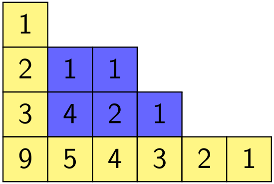
$$d_\lambda^+ = D(\lambda)^{1+O(\frac{1}{\ln n})},$$

and

$$d_\lambda^+ = d_\lambda^{1-O(\frac{1}{\ln n})}.$$

Done by induction on the size of the diagram.

The induction property



$$d_\lambda^+ = \binom{n}{|c|} d_s^+ d_c^+ .$$

Conclusion

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Theorem [Teyssier & T. '24]

For all $n \geq 1$, let $\mathcal{C}_n$ be a conjugacy class of $\mathfrak{S}_n$ without fixed point. Then, the mixing time for $(\mathcal{C}_n)_{n\geq 1}$ is 2 if and only if the number of transpositions in $\mathcal{C}_n$ is $o(n)$.

Random maps and surfaces

How to build a random surface?

1) Start from k polygons of respective perimeters $(p_1, \dots, p_k)$.
2) Label their sides from 1 to $N := \sum_k p_k$, uniformly at random.
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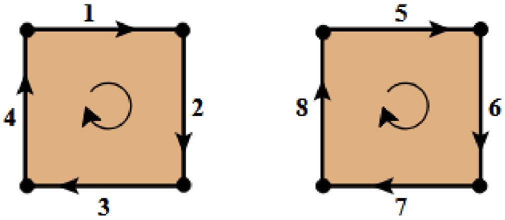
Coded by two permutations :

- $\alpha \in \mathfrak{S}_N$, where $\alpha(i) = j$ if the side j 'follows' the side i around a polygon.
- $\beta \in \mathfrak{S}_N$, where $\beta(i) = j$ if the sides i and j are glued. β is a fixed-point-free involution (product of $N/2$ transpositions).

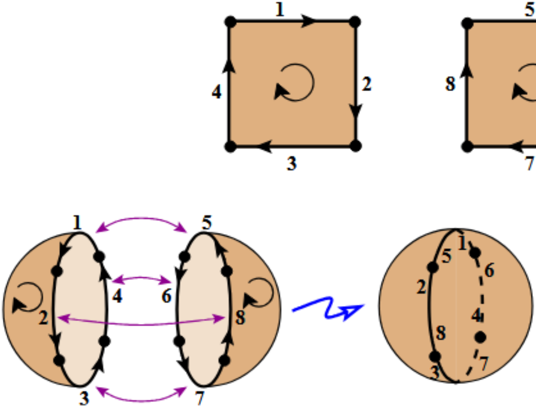
Permutation α



$$\alpha = (1234)(5678)$$

[Chmutov]

Permutation β



$$\beta = (15)(28)(37)(46)$$

Our results

Let $\mathcal{P}_N$ be a sequence of $k(N)$ polygons with N sides in total, where

- each polygon has *at least* 2 sides ;
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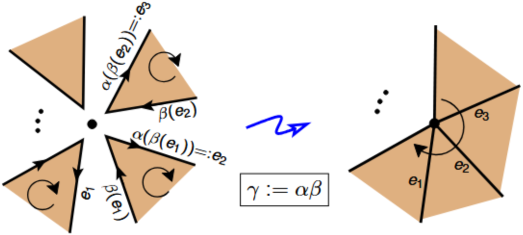
- Completes results of [Gamburd '06], [Chmutov & Pittel '16], [Budzinski, Curien & Petri '19].

Proof

- α, β are fixed-point-free, et α has $o(N)$ transpositions.
- Our previous results can be adapted: the product $\gamma := \alpha \circ \beta$ is asymptotically uniform.
- A cycle of γ corresponds to a vertex of M_n.



Thanks!