Cortipom25, Cortipom ANR project
Le Croisic, June 9-13, 2025

The Tangent Method: an overview

(joint work with A Sportiello, 2016)

Filippo Colomo
INFN, Florence

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Bonus track:

Tracy-Widom in large ASMs

(joint work with A G Pronko, 2024)

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What is the Tangent Method?

- The Tangent Method is an exact, albeit heuristic recipe to derive the explicit analytic expression of arctic curves in models *that can be formulated in terms of directed lattice paths.*
  NB: we are NOT restricting to NILP; osculating paths OK.
- This is a fairly general condition, since arctic curves and limit shapes usually follow from some discrete height function, whose level curves, under some weak monotonicity condition, may be viewed as a set of directed lattice paths on the underlying lattice.
- Originally devised in the context of the six-vertex model, the method has been then successfully applied to many different situations where a lattice path description is available.
- The six-vertex model being a rather complicate model, we shall here present the method on a simplified version of it (although still non-determinantal, and far from trivial), namely Alternating Sign Matrices.

Alternating Sign Matrices (ASMs)

An ASM [Mills-Robbins-Rumsey'83] is a square matrix of 0, 1, -1, such that:

- in each row and column entries alternate in sign;
- for any given row or column, entries sum up to 1.

$$\text{A } 5 \times 5 \text{ ASM :} \qquad \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

NB: only a single nonzero entry in the first (last) row (column).

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e.g, $N = 3$:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

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- Enumeration: $A_N := \#$ ASMs of size N *(partition function)*
- Refined enumeration: $A_{N,r} := \#$ ASMs of size N with their sole non-zero entry in the first row at position r *(refined partition function)*
- *boundary correlation function*: $H_{N,r} := A_{N,r}/A_N$,

  and its generating function: $$h_N(z) := \sum_{r=1}^{N} H_{N,r} z^{r-1}, \qquad h_N(1) = 1.$$

Alternating Sign Matrices (ASMs)

e.g, $A_3 = 7$:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

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$A_{3,1} = 2,\qquad A_{3,2} = 3,\qquad A_{3,3} = 2,\qquad h_3(z) = \frac{2}{7} + \frac{3}{7}z + \frac{2}{7}z^2$

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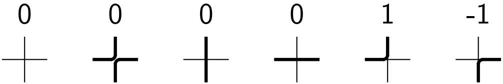
- $A_N = \prod_{j=1}^{N} \frac{(3j-2)!}{(2N-j)!}$ [Zeilberger'94] [Kuperberg'96]
- $A_{N,r} = \binom{N+r-2}{N-1}\binom{2N-1-r}{N-1}\binom{3N-2}{N-1}^{-1} A_N$ [Zeilberger'96]
- $h_N(z) = \frac{(N)_{N-1}}{(2N)_{N-1}}\, {}_2F_1\left(\left.\begin{matrix}-N+1,\ N\\ -2N+2\end{matrix}\right| z\right)$

ASMs as osculating lattice path

Robbins-Rumsey'86][Kuperberg'96]



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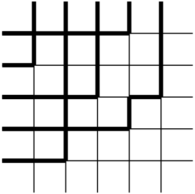

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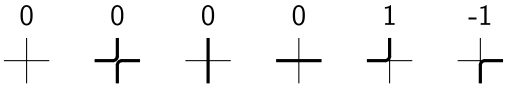

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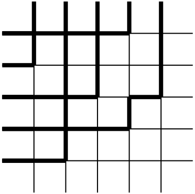
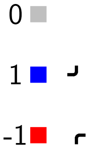
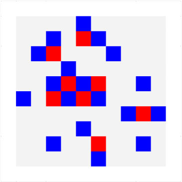
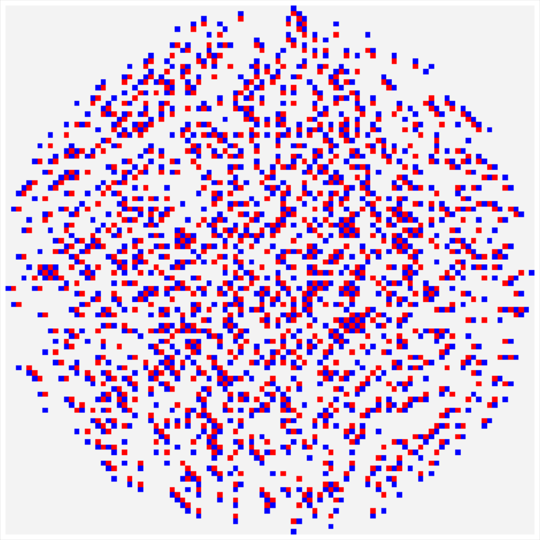




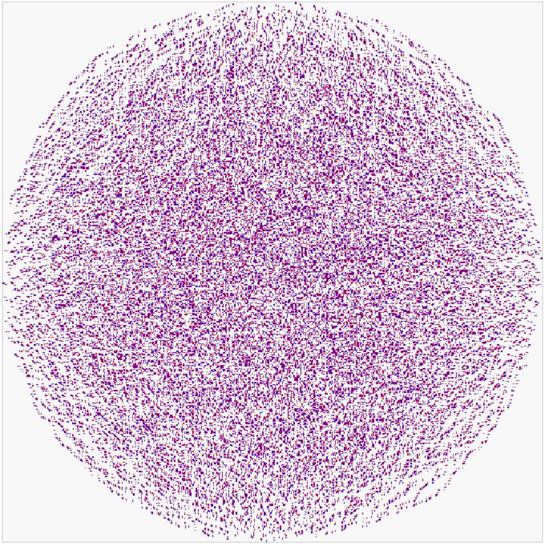


A 10×10 ASM

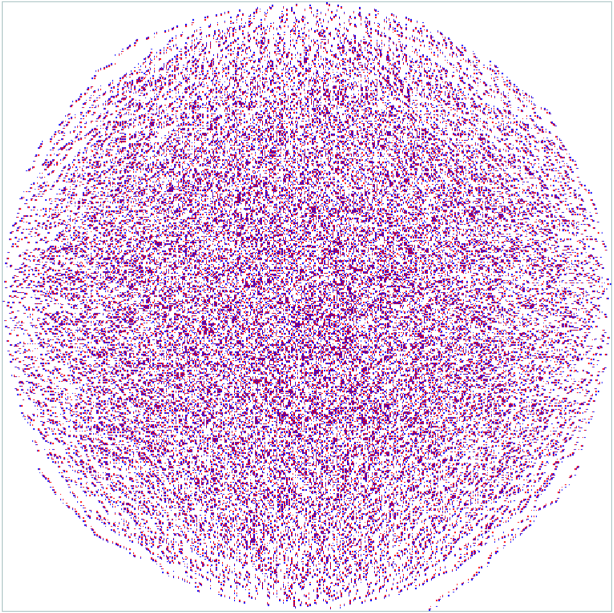
Here and below, numerics produced with improved version of a C code, kindly provided by Ben Wieland, based on the 'Coupling From The Past' [Propp-Wilson'96]



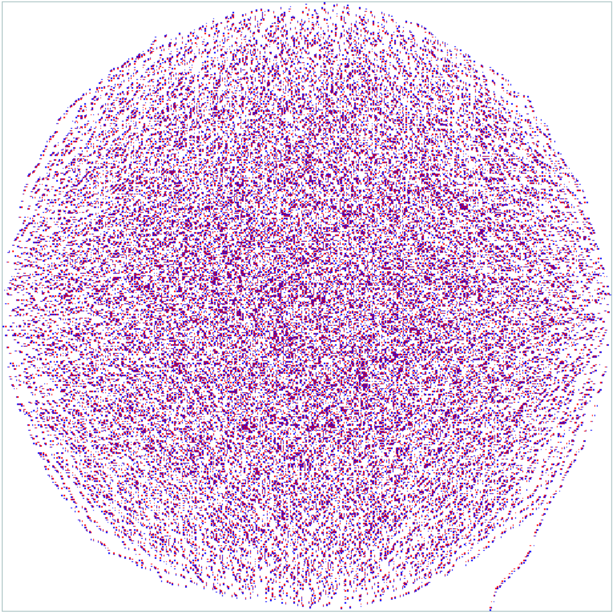
A 100×100 ASM



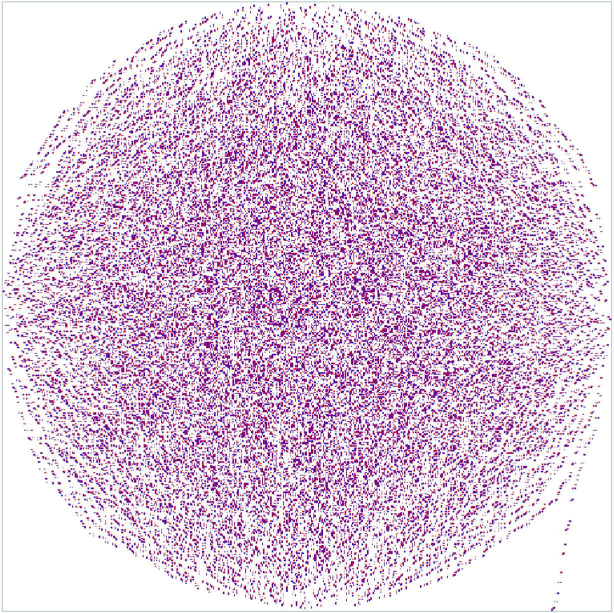
A 500×500 black ASM



A 500×500 ASM refined at position 350



A 500×500 ASM refined at position 400



A 500×500 ASM refined at position 450

The Tangent Method: the idea

Consider an $N \times N$ ASM refined at position r . Assume that:

1. In the scaling limit, the arctic curve of the first $N-1$ paths is exactly the same as that of an unrefined ASM.
2. When the Nth path first reach a location at a distance $\mathcal{O}(N^{\frac{1}{2}})$ from the $(N-1)$th path, then its remaining portion, from there till the conditioned location (N, r), is almost surely a random directed lattice path. As such, in the scaling limit it turns into a straight line.
3. In the scaling limit the obtained straight line *departs tangently from the original limit shape*.

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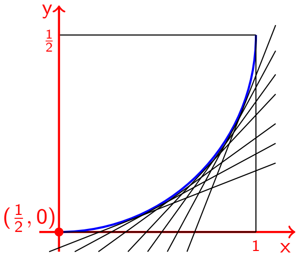
Rescale $i = \lceil (1-y)N \rceil$ and $j = \lceil xN \rceil$. Under above assumptions we expect:



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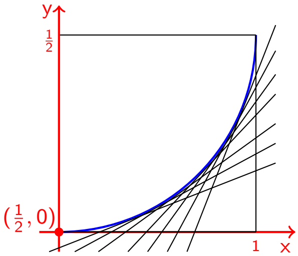
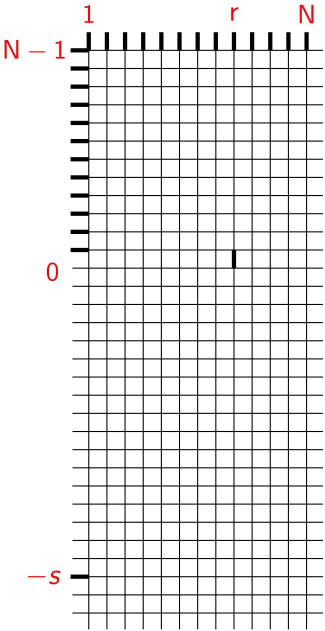
3. In the scaling limit the obtained straight line *departs tangently from the original limit shape*.

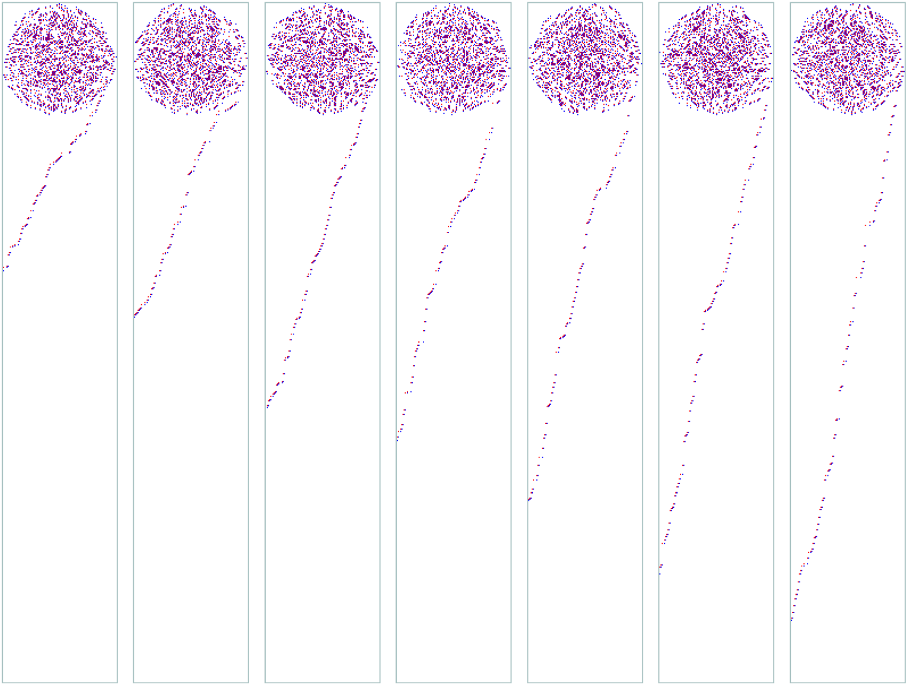
Rescale $i = \lceil (1-y)N \rceil$ and $j = \lceil xN \rceil$. Under above assumptions we expect:

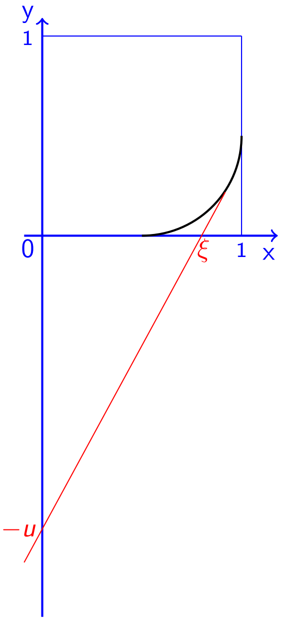
The arc of Arctic curve subtended by a given corner is the geometric caustic (or envelope) of the family of straight lines generated by varying the position of the refinement.

But how do you compute the slope?




1
r
N
N − 1
0
−s





Recall, $i = \lceil xN \rceil$, and $j = \lceil (1-y)N \rceil$.

Let $r = \lceil \xi N \rceil$, with $\xi \in (0,1)$,

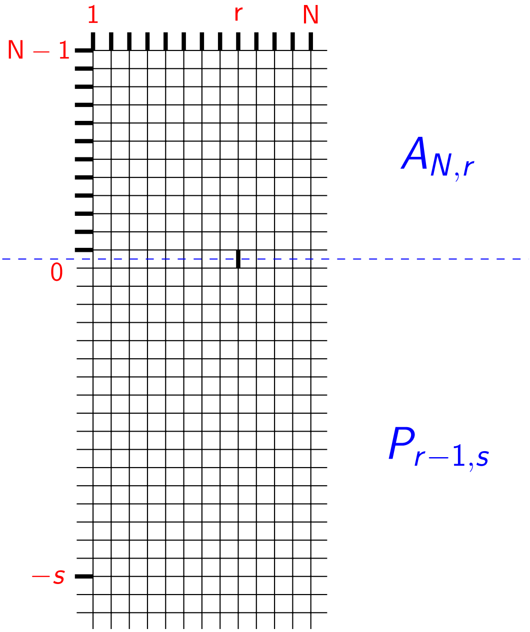
and $s = \lceil uN \rceil$, with $u \in (0,\infty)$.

The line through $(0,-u)$ and $(\xi, 0)$ has equation

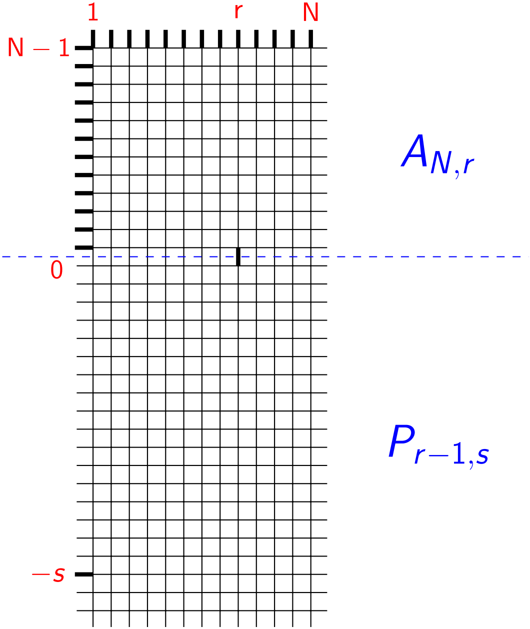
$$\frac{y+u}{x} = \frac{y}{x-\xi}$$

i.e

$$\boxed{x - \frac{\xi(u)}{u} y - \xi(u) = 0, \qquad u \in (0,\infty).}$$



1
r
N
N − 1
0
−s
$A_{N,r}$
$P_{r-1,s}$



$$Z_{N,s} = \sum_{r=1}^{N} A_{N,r} P_{r-1,s}$$

A digression on function $h_N(z)$

Define:

$$r(z) := \lim_{N\to\infty} \frac{1}{N} z \frac{\mathrm{d}}{\mathrm{d}z} \log h_N(z),$$

which will play a crucial role below.
Recall,

$$h_N(z) := \sum_{r=1}^{N} H_{N,r} z^{r-1},$$

with $H_{N,r}$ log-concave. Letting $r = \lceil \xi N \rceil$, with $\xi \in (0,1)$, at large N we may write

$$h_N(z) \propto \int_0^1 H_{N,\lceil \xi N \rceil} \mathrm{e}^{\lfloor \xi N \rfloor \log z} \mathrm{d}\xi.$$

Then, in the saddle-point approximation, we have simply

$$\boxed{r(z) = \xi_{\mathrm{sp}},}$$

with ξ_{sp} solution of the saddle-point eq.

$$\boxed{\frac{1}{N} \frac{\mathrm{d}}{\mathrm{d}\xi} \log H_{N,\lceil \xi N \rceil} + \log z = 0.}$$

Back to the extended domain

Let $r = \lceil \xi N \rceil$, $\xi \in (0,1)$, and $s = \lceil uN \rceil$, $u \in (0,\infty)$. We are interested in minimising the 'free energy'

$$F(u) := -\lim_{N\to\infty} \frac{1}{N} \log \frac{Z_{N,s}}{Z_{N,0}}$$

where

$$\frac{Z_{N,s}}{Z_{N,0}} = \sum_{r=1}^{N} H_{N,r} P_{r-1,s} = \sum_{r=1}^{N} H_{N,r} \binom{r+s-1}{r-1},$$

Saddle-point approximation: introduce the 'action'

$$\frac{1}{N} \log H_{N,\xi N} + \ell(\xi+u) - \ell(\xi) - \ell(u), \qquad \ell(x) := x \log x$$

whose saddle-point equation reads

$$\boxed{\frac{1}{N} \frac{\mathrm{d}}{\mathrm{d}\xi} \log H_{N,\xi N} + \log(\xi+u) - \log \xi = 0}$$

Summing up

From the very definition $\quad r(z) := \lim_{N\to\infty} \frac{1}{N} z \frac{\mathrm{d}}{\mathrm{d}z} \log h_N(z), \quad$ it follows that:

$$r(z) = \xi_{\mathrm{sp}}, \qquad \text{with} \qquad \frac{1}{N}\frac{\mathrm{d}}{\mathrm{d}\xi} \log H_{N,\lceil \xi N \rceil} + \log z = 0.$$

Moreover, for any given $u \in (1, \infty)$, the SPE for the $F(u)$,

$$\frac{1}{N}\frac{\mathrm{d}}{\mathrm{d}\xi} \log H_{N,\xi N} + \log(\xi + u) - \log \xi = 0$$

determines the value $\xi(u)$ and hence where $r = \lceil \xi N \rceil$ concentrates. It follows that:

$$z = \frac{\xi(u) + u}{\xi(u)} \qquad \Rightarrow \qquad \frac{u}{\xi(u)} = z - 1$$

Insert in the eq. for the family of lines, and get:

$$\boxed{x - \frac{1}{z-1} y - r(z) = 0, \qquad z \in (1, \infty)}$$

Some comments

$$x - \frac{1}{z-1}y - r(z) = 0, \qquad z \in (1, \infty)$$

- This eq. is quite general. Not only ASMs. For different lattices, domain shapes, coordinates, minimal changes required. Directly applicable to your favourite model of directed lattice path (with uniform weights).
- Extension to non-uniform weights is fairly straightforward.
- All the information on the original model is inside $r(z)$. Of course, this need to be evaluated explicitely, which may be difficult.
- *'Holographic principle"*, this eq. allows you to calculate 'bulk' quantities (the limit shape) just from boundary knowledge, $r(z)$.
- Arctic curve (parametric form):

$$\begin{cases} x - \frac{1}{z-1}y - r(z) = 0 \\ y - (z-1)^2 r'(z) = 0 \end{cases} \Rightarrow \begin{cases} x(z) = (z-1)r'(z) + r(z) \\ y(z) = (z-1)^2 r'(z) = 0 \end{cases} \quad z \in (1, \infty)$$

And for $z \notin (1, \infty)$?

Back to ASMs

We need to evaluate $r(z) := \lim_{N\to\infty} \frac{1}{N} z \frac{\mathrm{d}}{\mathrm{d}z} \log h_N(z)$,

where $$h_N(z) = \frac{(N)_{N-1}}{(2N)_{N-1}} \, {}_2F_1\left(\left.\begin{matrix} -N+1,\ N \\ -2N+2 \end{matrix}\right| z\right).$$ [Zeilberger'96]

A simple saddle-point evaluation yields: $$r_{ASM}(z) = \frac{\sqrt{z^2 - z + 1} - 1}{z - 1}$$

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$$\begin{cases} x(z) = \frac{2z-1}{2\sqrt{z^2-z+1}} \\ y(z) = 1 - \frac{z+1}{2\sqrt{z^2-z+1}} \end{cases} \qquad z \in (1, \infty),$$

derived in [FC-Sportiello'16]
reproduces [FC-Pronko'09]
proven rigorously in [Aggarwal'19]

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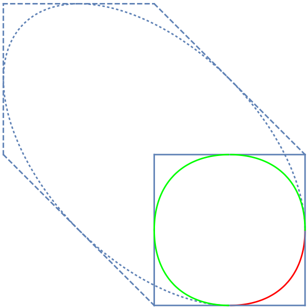

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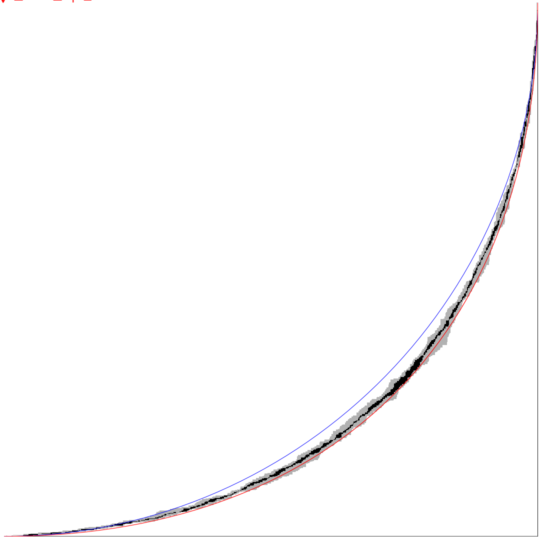


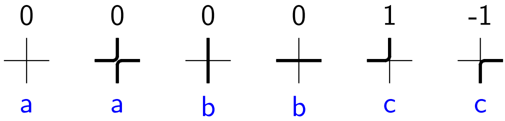
$\Delta = \frac{1}{2}$

$\Delta = 0$

Picture by Ben Wieland, `https://nokedli.net/asm-frozen/`

Six-vertex model (with Domain Wall BC)

[Lieb'67] [Sutherland'67] [Korepin'82]



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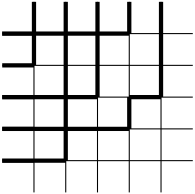

Weighted, and refined weighted enumerations may be defined just as for ASMs, but they now depend on a, b, c.

$$A_N \ \rightarrow \ Z_N(a, b, c) := \sum_{conf} a^{n_a} b^{n_b} c^{n_c}$$

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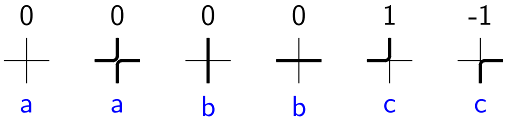

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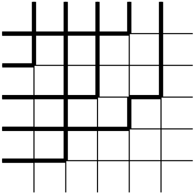


Formulae become quite bulky, e.g., $P_{(r,s)} = \left(\frac{b}{a}\right)^{r+s} \sum_{l \leq 0} \binom{r}{l}\binom{s}{l} \left(\frac{c}{a}\right)^{2l+1}$

and the family of lines reads:

$$x - \frac{zc^2}{(z-1)(b^2 z - 2ab + a^2)} y - r(z) = 0, \qquad z \in (0, \infty)$$

Calculation of $r(z)$ requires some work too [FC-Pronko'09].

Arctic curves ($a = b = 1, \quad c = \sin 2\eta$)

$$x = F\left(\frac{\pi}{2} - \eta - \zeta\right) \qquad \zeta \in [0, \frac{\pi}{2} - \eta]$$

$$y = F(\zeta)$$

where

$$F(\zeta) = \frac{\sin^2\zeta \sin^2(\zeta + 2\eta)\cos(\zeta - \eta)\cos(\zeta + \eta)}{\sin 2\eta \cos\eta \big[\cos(\zeta - \eta)\sin\zeta + \cos(\zeta + \eta)\sin(\zeta + 2\eta)\big]}$$

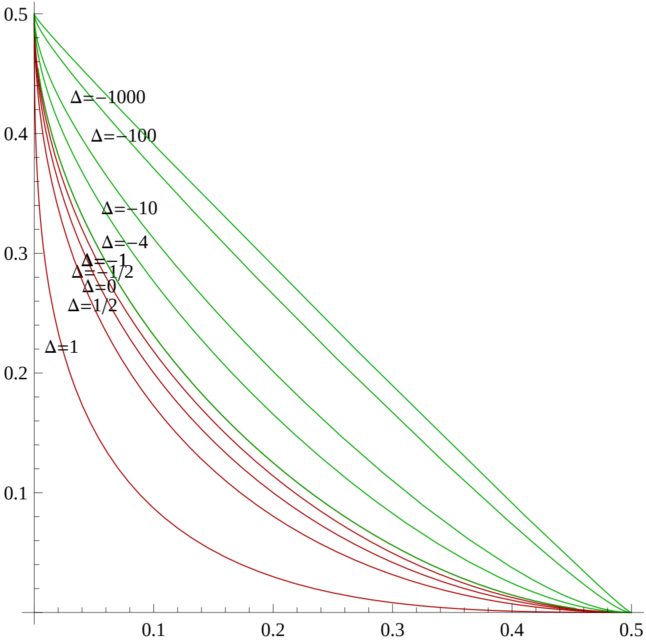
$$\times \left\{ \frac{\cos^2\eta}{\sin^2\zeta \cos(\zeta + \eta)\cos(\zeta - \eta)} \right.$$

$$- \frac{\sin 2\zeta}{\cos(\zeta - \eta)\cos(\zeta + \eta)} \frac{\alpha \sin\alpha(\frac{\pi}{2} - \eta)}{\sin\alpha\zeta \sin\alpha(\zeta + \frac{\pi}{2} - \eta)}$$

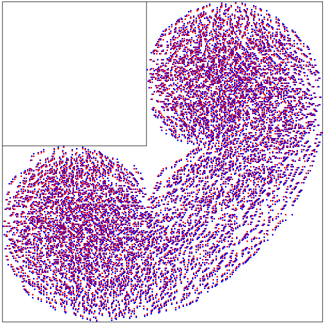
$$\left. - \frac{\alpha^2 \sin\alpha(2\zeta + \frac{\pi}{2} - \eta)\sin\alpha(\frac{\pi}{2} - \eta)}{\sin^2\alpha\zeta \sin^2\alpha(\zeta + \frac{\pi}{2} - \eta)} \right\}.$$

NB: $\alpha = \frac{\pi}{\pi - \arccos\left(1 - \frac{c^2}{2}\right)}, \qquad \eta = \frac{1}{2}\arccos\left(1 - \frac{c^2}{2}\right)$

Arctic curves ($a = b = 1, \quad \Delta := 1 - \frac{c^2}{2}$)



Other domain shapes?



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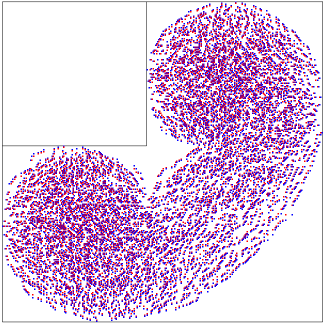
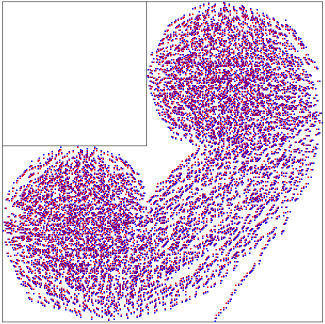
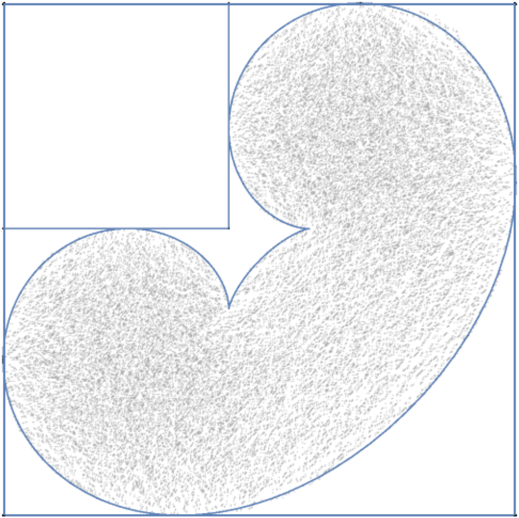




Apply the Tangent Method!

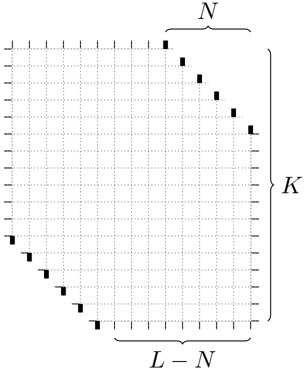
Domino Tiling of AD with a cut-off corner ($c = \sqrt{2}$)

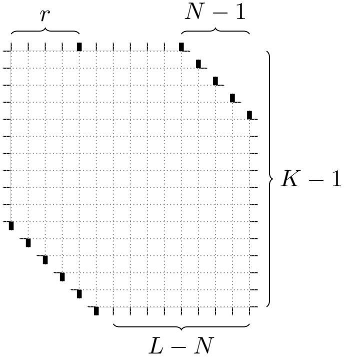
[FC-Pronko-Sportiello'18]



NILP? Easy!

Exercise:





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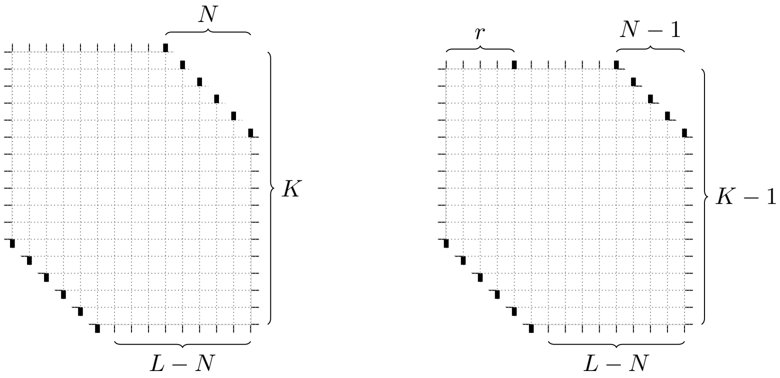


Apply [Linstrom'73][Gessel-Viennot'85] to express both partition functions as determinants of matrice binomials. Then use Advanced Determinant Calculus, Thm. 26 [Krattenthaler'99], to express the ratio as product of factorials. Use Stirling to evaluate asymptotics and obtain the arctic curve.

NB: extend the domain upward, and modify coordinates accordingly!

Other models?

- Domino tilings of Aztec Diamond [Jockusch-Propp-Shor'98] [FC-AS'16]
- Boxed plane partitions [Cohn-Larsen-Propp'98] [FC-AS'16]
- ASM's on a triangoloid domain [FC-AS'16] [Aggarwal'19]
- Twenty-vertex model [Debin-Di Francesco-Guitter'20]
- Twenty-vertex model on a triangle [Di Francesco'23]
- Aztec rectangles with defects [Di Francesco-Guitter'19]
- Double Aztec rectangles [de Kemmeter-Debin-Ruelle'22]
- Two-periodic Aztec Diamond [Chhita-Johansson'16][Duits-Kuijlaars'21] [Ruelle'22]
- Dimer models with defects and freezing boundary [Debin-Ruelle'19]
- Improvements of the method (tangency) [Debin-Granet-Ruelle'19]
- Multirefinements [Debin-Ruelle'21]
- q-weighted paths [Di Francesco-Guitter'19]
- Octahedron equation [Di Francesco-Soto-Garrido'19]
- Bounded Lecture Hall Tableaux [Corteel-Keating-Nicoletti'20] ...

INTERMISSION

Cortipom25, Cortipom ANR project
Le Croisic, June 9-13, 2025

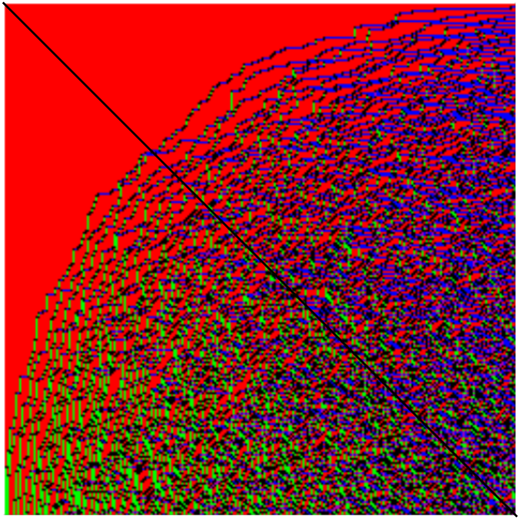
Bonus track:

Tracy-Widom in large ASMs

(joint work with A G Pronko, 2024)

Filippo Colomo
INFN, Florence

Interface fluctuations



Square Emptiness Formation Probability (SEFP)

Let $\mathcal{A}_N$ be the set of all ASMs of size N.

Let $\mathcal{B}_{N,s} := \{m \in \mathcal{A}_N, m_{i,j} = 0 \ \forall i, j \leq s\}$

An example of matrix in $\mathcal{B}_{7,3}$:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

For given N, s, sample M uniformely from $\mathcal{A}_N$. We call SEFP, and denote by $F_{N,s}$ the probability to get $M \in \mathcal{B}_{N,s}$. Clearly,

$$F_{N,s} := \frac{|\mathcal{B}_{N,s}|}{|\mathcal{A}_N|}$$

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- discriminates the transition between top-left empty (or frozen) region and central non-empty (disordered) region
- in the scaling limit: stepwise behaviour in correspondence of the Arctic curve
- with some smoothing at some scale N^α, $0 < \alpha < 1$, to be determined

Multiple Integral Representation for $F_{N,s}$

NB: this is a theorem! [FC-Pronko'08]

The following representation holds

$$F_{N,s} = \oint_{C_\infty} \cdots \oint_{C_\infty} J_N^{(s)}(z_1, \ldots, z_s) \mathrm{d}^s z$$

where

$$J_N^{(s)}(z_1, \ldots, z_s) = \frac{1}{(2\mathrm{i}\pi)^s} \prod_{j=1}^{s} \frac{1}{z_j^j (z_j - 1)^{s-j+1}} \prod_{1 \le j < k \le s} \frac{z_j - z_k}{z_j z_k - z_k + 1} h_{N,s}(z_1, \ldots, z_s)$$

with

$$h_{N,s}(z_1, \ldots, z_s) := \frac{1}{\Delta_s(z_1, \ldots, z_s)} \det \left[(z_j - 1)^k z_j^{s-k} h_{N-s+k}(z_j) \right]_{j,k=1}^{s}$$

and

$$h_N(z) = \frac{(N)_{N-1}}{(2N)_{N-1}} \, {}_2F_1 \left(\begin{matrix} -N+1, \, N \\ -2N+2 \end{matrix} \middle| z \right).$$

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$$|\mathcal{A}_5| = A_5 = 429$$

$$|\mathcal{B}_{5,2}| = 96$$

$$F_{5,2} = \frac{32}{143}$$

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and

$$h_N(z) = \frac{(N)_{N-1}}{(2N)_{N-1}} \, {}_2F_1 \left(\left. \begin{matrix} -N+1, \; N \\ -2N+2 \end{matrix} \right| z \right).$$

$$|\mathcal{A}_{14}| = A_{14} = 9995541355448167482000 \simeq 10^{22}$$

$$|\mathcal{B}_{14,5}| = 1184591399320711517 2 \simeq 10^{19}$$

$$F_{14,5} = \frac{3870965779057}{3266307568354500}$$

Determinant Representation for $F_{N,s}$

NB: this is a conjecture! [FC-Pronko'24]

The following representation holds

$$F_{N,s} = \det_s (I - A)$$

where the $s \times s$ matrix $A = A(N, s)$ reads

$$A_{ij} = \oint_{C_0} \oint_{C_0} \frac{e_i^L(z) e_j^U(w)}{1 - z - w} \frac{\mathrm{d}z \mathrm{d}w}{(2\pi \mathrm{i})^2}, \qquad i, j = 1, \ldots, s, \qquad (*)$$

with

$$e_i^L(z) := \frac{(1-z)^{i-1}}{z^i} \left(1 + (-1)^i z\right) h_{N-s+i}(z),$$

$$e_j^U(w) := \frac{(1-w)^{j-1}}{h_{r+j}(0) w^j} \left(1 + (-1)^{j+1} w\right) h_{N-s+j}(w).$$

Built analitically from previous MIR for $s = 1, 2, 3, 4$ and conjectured to hold for all integer s.

Check

Check the $s = 5$ case: evaluate with Mathematica both our conjectural expression and the MIR, for $N = 7, \ldots, 13$:

N	*Determinant*	*MIR*
7	0	0
8	0	0
9	0	0
10	$\frac{61347}{43178090900}$	$\frac{61347}{43178090900}$
11	$\frac{49711519}{1636618150125}$	$\frac{49711519}{1636618150125}$
12	$\frac{54886057499}{221251085257500}$	$\frac{54886057499}{221251085257500}$
13	$\frac{3870965779057}{3266307568354500}$	$\frac{3870965779057}{3266307568354500}$

Fredholm determinant representation

The following representation holds [FC-Pronko'24]

$$F_{N,s} = \det(1 - \hat{K}_{[0,\infty)})$$

where $\hat{K}_{[0,\infty)}$ is a linear integral operator acting on functions defined on $\mathbb{R}^+$ according to the rule

$$(\hat{K}_{[0,\infty)} f)(t_1) = \int_0^\infty K(t_1, t_2) f(t_2) \mathrm{d}t_2$$

with kernel

$$K(t_1, t_2) = \oint_{C_0} \oint_{C_0} \mathrm{e}^{\left(z - \frac{1}{2}\right)t_1 + \left(w - \frac{1}{2}\right)t_2} \sum_{j=1}^{s} e_j^L(z) e_j^U(w) \frac{\mathrm{d}z \mathrm{d}w}{(2\pi \mathrm{i})^2}.$$

Remark

The kernel $K(t_1, t_2)$ is not 'of integrable form'

(in the sense of [Its-Izergin-Korepin-Slavnov'92])

Scaling limit

Define

$$F(\sigma) := \lim_{N\to\infty} \left(\det_s(1-A)\Big|_{s=N\left(1-\frac{\sqrt{3}}{2}\right)-\frac{N^{1/3}}{2^{4/3}3^{1/6}}\sigma} \right)$$

where the $s \times s$ matrix A is given by $(*)$.

Let $\hat{K}^{\mathrm{Ai}}_{[\sigma,\infty)}$the linear integral operator acting on space $L^2(\sigma,\infty)$ with kernel

$$K^{\mathrm{Ai}}(t_1,t_2) = \frac{\mathrm{Ai}(t_1)\,\mathrm{Ai}'(t_2) - \mathrm{Ai}'(t_1)\,\mathrm{Ai}(t_2)}{t_1 - t_2} \qquad \text{(Airy kernel)}$$

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Theorem (FC-Pronko'24)

The following holds:

$$F(\sigma) = \det\left(1-\hat{K}^{\mathrm{Ai}}_{[\sigma,\infty)}\right) =: \mathcal{F}_2(\sigma)$$

Conclusions

Conjecture

For the Square Emptiness Formation probability for ASMs, the following determinant representation holds:

$$F_{N,s} = \det_s(1 - A)$$

where $A = A(N, s)$ *is the* $s \times s$ *matrix given in* $(*)$.

Theorem

Given the $s \times s$ *matrix* $A = A(N, s)$, *see* $(*)$, *the following holds:*

$$\lim_{N \to \infty} \left(\det_s(1 - A) \Big|_{s = N\left(1 - \frac{\sqrt{3}}{2}\right) - \frac{N^{1/3}}{2^{4/3} 3^{1/6}} \sigma} \right) = \mathcal{F}_2(\sigma).$$

- First exact (although non rigorous) derivation of TW in a *critical* **and** *interacting* (in the sense of non-determinantal) system.
- In full agreeement with numerical simulations [Korepin-Lyberg-Viti'23] [Prauhofer-Spohn'24]
- In full agreement with the GOE behaviour derived for the fluctuations of TSSCPP [Ayyer,Chhita,Johansson'23]