Chow polynomials and Friends

Elena Hoster
Ruhr University Bochum

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Matroids

•) A MATROID $\mathcal{M}$ is a pair $(E, \mathrm{rk}: 2^E \longrightarrow \mathbb{Z}_{\geq 0})$ such that

ground set, finite

rank function

(1) $\mathrm{rk}(A) \leq |A|$ for $A \subset E$

(2) $\mathrm{rk}(A) \leq \mathrm{rk}(B)$ for $A \subseteq B \subseteq E$

(3) $\mathrm{rk}(A \cup B) + \mathrm{rk}(A \cap B) \leq \mathrm{rk}(A) + \mathrm{rk}(B)$

•) The RANK of $\mathcal{M}$ is $\mathrm{rk}(\mathcal{M}) := \mathrm{rk}(E)$.

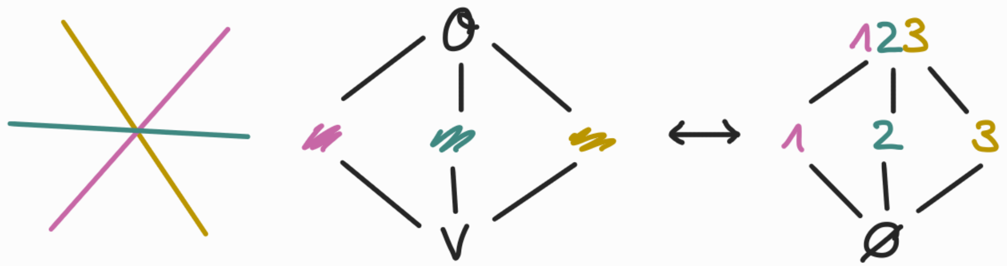
•) $F \subseteq E$ is a FLAT if $\mathrm{rk}(F \cup \{x\}) > \mathrm{rk}(F)$ for all $x \in E \setminus F$

•) $\mathcal{L}(\mathcal{M}) = (\{\text{flats}\}, \subseteq)$ the LATTICE OF FLATS is a geometric lattice of rank $\mathrm{rk}(\mathcal{M})$.

Examples

1) (central) Hyperplane Arrangements

lattice of intersections of hyperplanes $\longleftrightarrow$ lattice of flats



2) Uniform matroid $U_{k,n}$

rank

size of groundset

$E = [n] = \{1, \dots, n\}$

$\mathrm{rk}(A) = \min\{\#A, k\}$

$\mathcal{L}(U_{k,n}): (\{A \subseteq E \mid \#A < k\} \cup \{E\}, \subseteq)$

Chow polynomial: as Hilbert series of the Chow Ring

(Feichtner - Yuzvinsky, 2004)

The Chow ring of a matroid $\mathcal{M}$ is a graded $\mathbb{Z}$-algebra

$$A = A(\mathcal{L}(\mathcal{M})) = A^0 \oplus A^1 \oplus \cdots \oplus A^r \qquad r+1 = rk(\mathcal{M})$$

and has Hilbert series

$$H_{\mathcal{M}}(x) = \sum_{k=0}^{r} \underbrace{\dim(A^k)}_{=:a_k} \cdot x^k \,,$$

CHOW POLYNOMIAL

Chow ring $A(\mathcal{L},\mathcal{G})$ defined for any
- $\mathcal{L}$: lattice
- $\mathcal{G}$: building set

$\boxed{?}$ What do we know about $a_0, \dots, a_r$?

Combinatorial properties of $H_{\mathcal{M}}(x)$:

[Adiprasito - Huh - Katz, 2018]
- √ unimodal $\quad a_0 \le a_1 \le \dots \le a_{\lfloor \frac{r}{2} \rfloor} = a_{\lceil \frac{r}{2} \rceil} \ge \dots \ge a_r$
- √ palindromic $\quad a_k = a_{r-k}$ for $k \le r/2$

[Ferroni - Matherne - Stevens - Vecchi, 2024]
- √ γ-positive $\quad H_{\mathcal{M}}(x) = \sum_{i=0}^{\lfloor \frac{r}{2} \rfloor} \gamma_i \, x^i (1+x)^{r-2i}, \quad \gamma_i > 0$

Chow polynomial : Combinatorial properties

The CHOW POLYNOMIAL of a matroid $\mathcal{M}$ of rank $\operatorname{rk}(\mathcal{M}) = r+1$ is the Hilbert series of its Chow ring $A = \bigoplus_{k\geq 0} A^k$,

$$H_{\mathcal{M}}(x) = \sum_{k\geq 0} \dim(A^k)\cdot x^k = \sum_{k=0}^{r} a_k \cdot x^k = \sum_{i=0}^{\lfloor r/2 \rfloor} \gamma_i \, x^i (1+x)^{r-2i}$$

$$\boxed{a_k^2 \geq a_{k-1}\cdot a_{k+1} \quad \text{for } 1 \leq k < r}$$

$$\begin{tikzcd}
 & \text{log-concavity ?} \arrow[r, Rightarrow] & \text{unimodality } \checkmark \\
\text{? real-rootedness} \arrow[ur, Rightarrow] \arrow[dr, Rightarrow, "+\text{ palindromic}"'] & & \\
 & \gamma\text{-positivity } \checkmark \arrow[uur, Rightarrow] \arrow[r, Rightarrow] & \text{palindromicity } \checkmark
\end{tikzcd}$$

Brändén-Vecchi, 2025
$H_{\mathcal{M}}(x)$ real-rooted for $\mathcal{M} = U_{n,k}$ uniform

Chow polynomial : Combinatorial formulas

$\mathcal{L} = \mathcal{L}(\mathcal{M})$ of rank r

Thm: [Feroni-Matherne-Stevens-Vecchi, 2024]

$$H_{\mathcal{M}}(x) = \sum_{\substack{\mathcal{C}=\{\hat{0}=C_1<\cdots<C_{k+1}=\hat{1}\} \\ \text{chain in } \mathcal{L}}} \overline{\chi}_{\mathcal{L},\mathcal{C}}(x)$$

$$\chi_{\mathcal{L}}(t) = \sum_{F\in\mathcal{L}} \mu(\hat{0},F)\cdot t^{\operatorname{rank}(\mathcal{L})-\operatorname{rank}(F)}$$

$$\overline{\chi}_{\mathcal{L}}(t) = \frac{\chi_{\mathcal{L}}(t)}{(t-1)}$$

$$\overline{\chi}_{\mathcal{L},\mathcal{C}}(t) = \prod_{i=1}^{k} \overline{\chi}_{[C_i,C_{i+1}]}(t)$$

Thm: [Stump, 2024]

Let λ be an R-labeling on $\mathcal{L}(\mathcal{M})$.
Then,

$$H_{\mathcal{M}}(x) = \sum_{\mathcal{C}} x^{\operatorname{des}(\lambda_{\mathcal{C}})}\cdot(x+1)^{r-1-2\cdot\operatorname{des}(\lambda_{\mathcal{C}})}$$

λ R-labeling

edge-labeling, s.t. every interval has unique max. chain w/ increasing labels

One proof of the γ-positivity

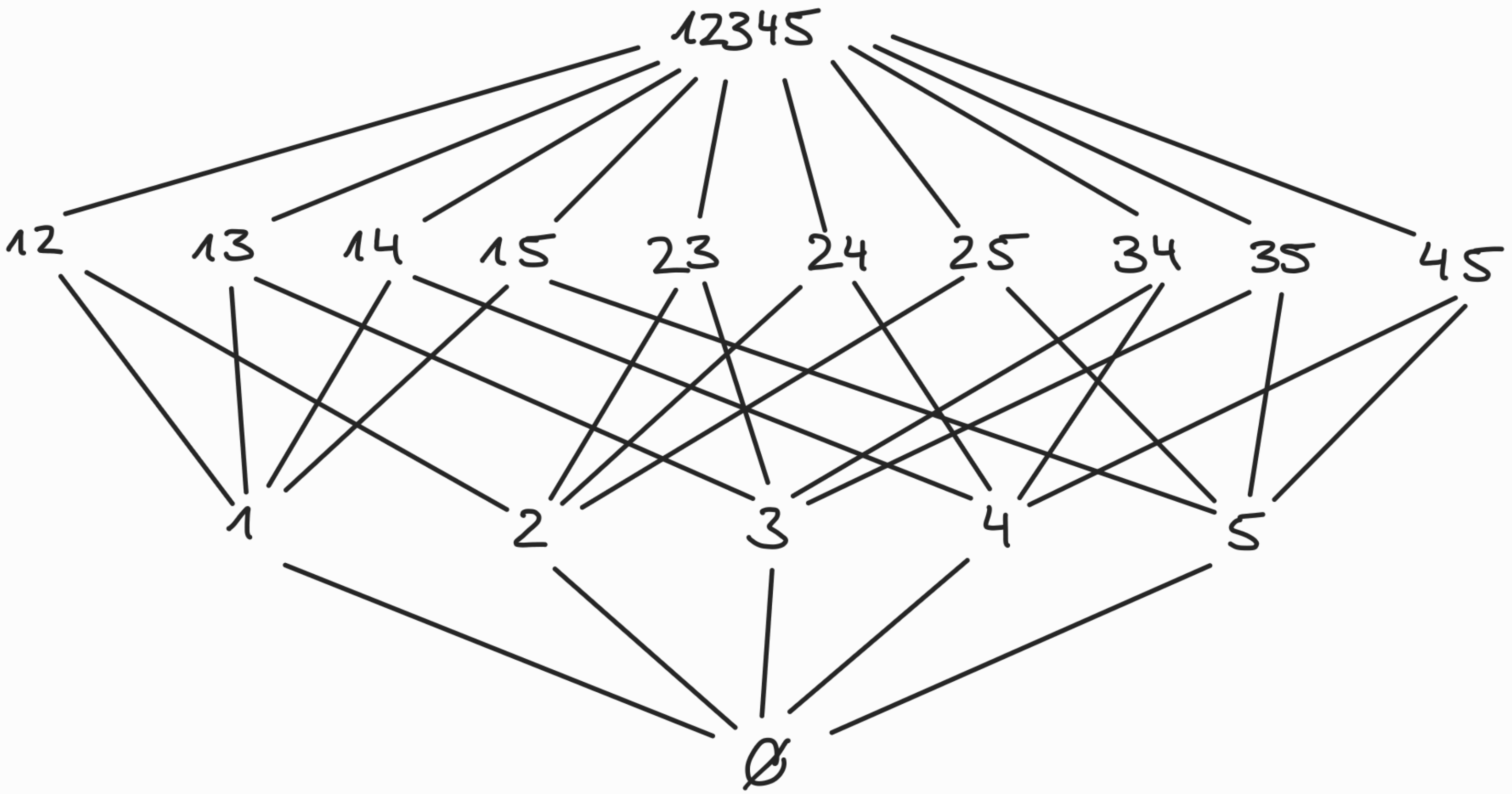
maximal chains $\mathcal{C}$ in $\mathcal{L}(\mathcal{M})$
s.t. for $\lambda_{\mathcal{C}}=(\lambda_1,\lambda_2,\dots,\lambda_k)$
- $\lambda_1<\lambda_2$
- $\lambda_{i-1}>\lambda_i \Rightarrow \lambda_i<\lambda_{i+1}$

Example : Chow polynomial of $U_{3,5}$

rank 3

on groundset $\{1,\dots,5\}$

$\mathcal{L}(U_{3,5})$:

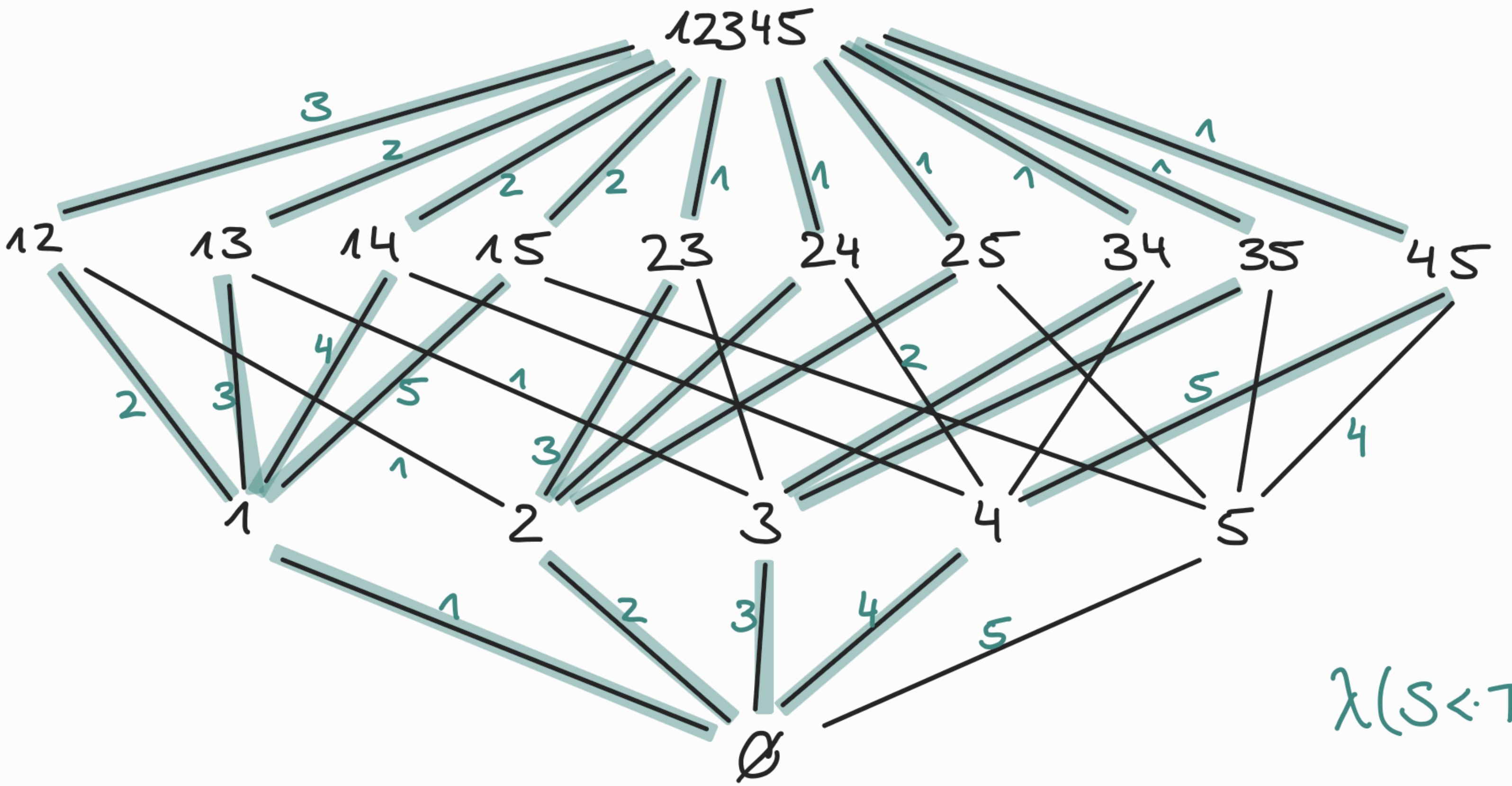


$$H_{U_{3,5}}(x) = \sum_{\substack{e=\{\hat{0}=c_1<\dots<c_{k+1}=\hat{1}\} \\ \text{chain in } \mathcal{L}}} \overline{\chi}_{\mathcal{L},e}(x) = [\dots] = 1 + 11x + x^2$$

Example : Chow polynomial of $U_{3,5}$

rank 3 — on groundset $\{1,\dots,5\}$

$\mathcal{L}(U_{3,5})$:



$\lambda(S \lessdot T) := \min(T \setminus S)$

$$H_{U_{3,5}}(x) = \sum_{e} x^{\operatorname{des}(\lambda_e)} \cdot (x+1)^{2-2\cdot \operatorname{des}(\lambda_e)} = 1 + 11x + x^2$$

Chow polynomial : of uniform matroids

Thm: [Ferroni-Matherne-Stevens-Vecchi, 2024]

Uniform matroids maximize Chow polynomials coefficient-wise :

$$H_{U_{k,n}}(x) - H_{\mathcal{M}}(x) \in \mathbb{N}[x]$$

for $\mathcal{M}$ matroid of rank k on a ground set of size n.

Thm [H., 2024+]

$$H_{U_{k,n}}(x) = \sum_{m=0}^{k-1} \#\left\{ \begin{array}{l} \text{loopless Schubert matroids} \\ \text{on ground set } [n] \text{ of rank } m+1 \\ \text{and cogirth} > n-k \end{array} \right\} \cdot x^m$$

$$= \sum_{\substack{D \subseteq \{2,\dots,k-1\} \\ i\in D \Rightarrow i+1 \notin D}} \#\{w \in \mathfrak{S}_n \mid \mathrm{Des}(w) = D\} \; x^{\#D} \cdot (1+x)^{k-1-2\cdot\#D}$$

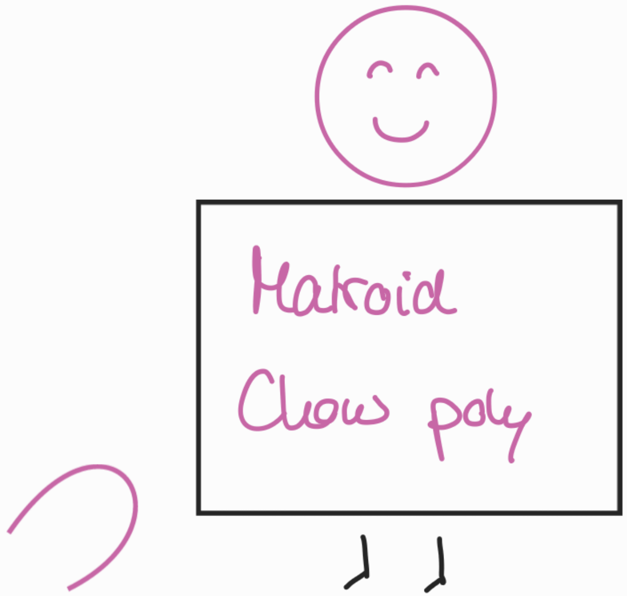
cogirth of $\mathcal{M}$
size of the smallest circuit of $\mathcal{M}^*$

? Can we write

$$H_{\mathcal{M}}(x) = \sum_{m\geq 0} \#\left\{ \begin{array}{l} \text{matroids on } [n] \text{ of rank } m+1 \\ \text{with } \boxed{\text{some restriction}} \end{array} \right\} \cdot x^m$$

$$H_{\mathcal{M}}(x) = \sum_{D} \#\left\{ w\in \mathfrak{S}_n \,\middle|\, \begin{array}{l} \mathrm{Des}(w) = D \text{ and} \\ \boxed{\text{some restriction}} \end{array} \right\} \cdot x^{|D|} \cdot (1+x)^{rk(\mathcal{M})-1-2\cdot|D|}$$

Chow polynomials & Friends



Hilbert-series
of Chow rings
$A(\mathcal{L}, \mathcal{G})$

Chow polynomial : of Posets

P a finite, graded, bounded poset.
The CHOW POLYNOMIAL of P is

$$H_P(x) = \sum_{\substack{\mathcal{C}=\{\hat{0}=C_1<\cdots<C_{k+1}=\hat{1}\} \\ \text{chain in } P}} \overline{\chi}_{P,\mathcal{C}}(x)$$

$$\chi_P(t) = \sum_{F\in P} \mu(\hat{0},F)\cdot t^{\mathrm{rank}(P)-\mathrm{rank}(F)}$$

$$\overline{\chi}_P(t) = \frac{\chi_P(t)}{(t-1)}$$

$$\overline{\chi}_{P,\mathcal{C}}(t) = \prod_{i=1}^{k} \overline{\chi}_{[c_i,c_{i+1}]}(t)$$

•) $H_{\mathcal{L}(M)}(x) = H_M(x)$
•) $H_P(x)$ unimodal & palindromic
•) $H_P(x)$ γ-positiv for P Cohen-Macaulay [Ferroni-Matherne-Vecchi]

? Conjecture: $H_P(x)$ real-rooted if P Cohen-Macaulay.

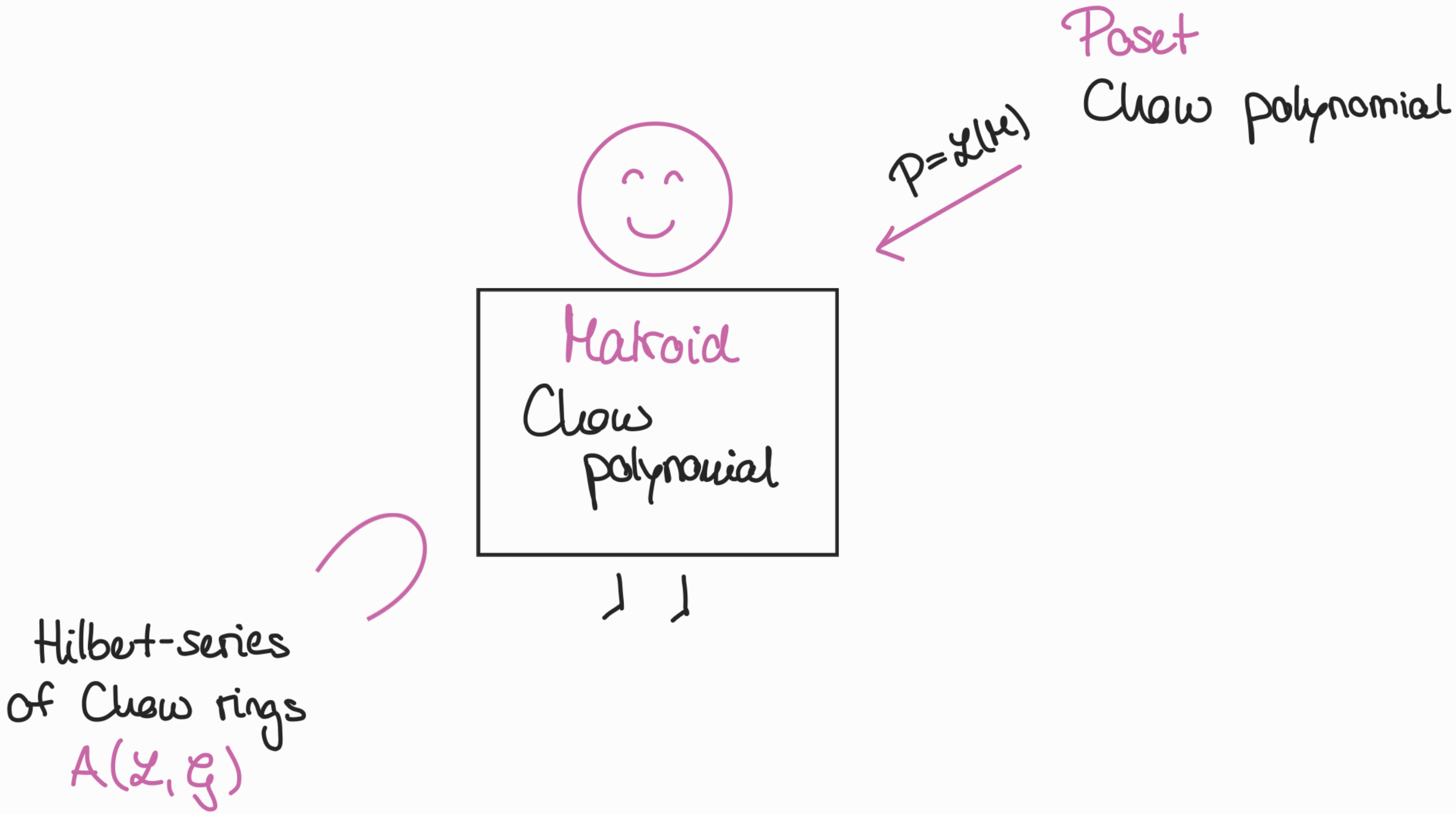
Thm: [Stump, 2024+]
If P admits $\mathcal{R}$-labeling λ, then

$$H_P(x) = \sum_{\mathcal{C}} x^{\mathrm{des}(\lambda_{\mathcal{C}})}\cdot(x+1)^{\mathrm{rk}(P)-1-2\cdot\mathrm{des}(\lambda_{\mathcal{C}})}$$

some maximal chains in P

P Cohen-Macaulay
$\Rightarrow P$ admits $\mathcal{R}$-labeling

Chow polynomials & Friends



Chow polynomials & Friends

Poset
Chow polynomial

$P = \mathcal{L}(M)$

$K = \chi$

Matroid
Chow polynomial

K - Chow function

Hilbert-series
of Chow rings
$A(\mathcal{L}, \mathcal{G})$

h-polynomial
of order complex

KL polynomials
of Coxeter groups

Extended ab-index

... of a finite, graded, bounded poset P of rank n is

$$\mathrm{ex}\Psi_P(y;a,b) = \sum_{\substack{e \text{ chain} \\ \text{in } P\setminus\{\hat{1}\}}} \underbrace{y^n \cdot \chi_{P,e}(-y^{-1})}_{\mathrm{Poin}_e(y)} \cdot \mathrm{wt}_e(a,b) \quad \in \mathbb{N}[y]\langle a,b\rangle$$

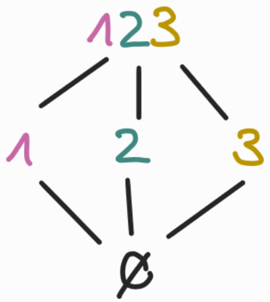
noncommuting

$$\boxed{\mathrm{wt}_e(a,b) = w_0 w_1 \cdots w_{n-1} \quad \text{where } w_i := \begin{cases} b, & i \in \mathrm{rk}(e) \\ a-b, & i \notin \mathrm{rk}(e) \end{cases}}$$

delete first letter in ab-words

$$\mathrm{ex}\tilde{\Psi}_P(y;a,b)$$

Example:



$e \cup \{\hat{1}\}$	$\mathrm{Poin}_e(y)$	$\mathrm{wt}_e(a,b)$
–	1	$(a-b)^2$
$\{\emptyset\}$	$1+3y+2y^2$	$b(a-b)$
$1,2,3$	$(1+y)$	$(a-b)b$
$\emptyset<1,2,3$	$(1+y)^2$	b^2

$$\Rightarrow \quad \mathrm{ex}\Psi(y;a,b) = (a-b)^2 + (1+3y+2y^2)\,b(a-b) + 3\cdot(1+y)(a-b)b + 3\cdot(1+y)^2 b^2$$
$$= a^2 + (3y+2y^2)ba + (2+3y)ab + y^2b^2$$

$$\Rightarrow \quad \mathrm{ex}\tilde{\Psi}(y;a,b) = a + (3y+2y^2)\,a + (2+3y)\,b + y^2 b = (1+3y+2y^2)a + (2+3y+y^2)b$$

Extended ab-index

... of a finite, graded, bounded poset P of rank n is

$$ex\Psi_P(y;a,b) = \sum_{\substack{e \text{ chain} \\ \text{in } P\setminus\{\hat{1}\}}} \underbrace{y^n \cdot \chi_{P,e}(-y^{-1})}_{\mathrm{Poin}_e(y)} \cdot \mathrm{wt}_e(a,b) \quad \in \mathbb{N}[y]\langle a,b\rangle$$

noncommuting

$$\boxed{\begin{aligned} &= w_0 w_1 \cdots w_{n-1} \\ &\text{where } w_i := \begin{cases} b &, i \in \mathrm{rk}(e) \\ a-b &, i \notin \mathrm{rk}(e) \end{cases}\end{aligned}}$$

delete first letter in ab-words

$$ex\tilde{\Psi}_P(y;a,b)$$

Let P be $\mathcal{R}$-labeled, then

$$\begin{tikzcd}
ex\tilde{\Psi}_P(y;a,b) \arrow[rr, mapsto, "\boxed{\begin{array}{l} y\mapsto -x \\ a\mapsto 1 \\ b\mapsto x\end{array}}"'] \arrow[d, mapsto, "\boxed{\begin{array}{l} a\mapsto 1 \\ b\mapsto t\end{array}}"] & & (1-x)^n \cdot H_P(x) \\
(1-t)^n \cdot \mathrm{cfHP}_P(y,t) \arrow[urr, mapsto, "\boxed{\begin{array}{l} y\mapsto -x \\ t\mapsto x\end{array}}"'] & &
\end{tikzcd}$$

coarse flag Hilbert Poincaré series

Extended ab-index

... of a finite, graded, bounded poset P of rank n is

noncommuting

$$\mathrm{ex}\Psi_P(y;a,b) = \sum_{e \text{ chain in } P} \underbrace{y^n \cdot \chi_{P,e}(-y^{-1})}_{\mathrm{Poin}_e(y)} \cdot \mathrm{wt}_e(a,b) \quad \in \mathbb{N}[y]\langle a,b\rangle$$

$$\boxed{\begin{aligned} &= w_0 w_1 \cdots w_{n-1} \\ &\text{where } w_i := \begin{cases} b, & i \in \mathrm{rk}(e) \\ a-b, & i \notin \mathrm{rk}(e) \end{cases}\end{aligned}}$$

$$\Big\downarrow \text{ delete first letter in ab-words}$$

$$\mathrm{ex}\tilde{\Psi}_P(y;a,b)$$

Example :

$$\begin{tikzcd}
(1+3y+2y^2)a + (2+3y+y^2)b \arrow[rr, mapsto, "\boxed{\begin{smallmatrix} y\mapsto -x \\ a\mapsto 1 \\ b\mapsto x\end{smallmatrix}}"'] \arrow[d, mapsto, "\boxed{\begin{smallmatrix} a\mapsto 1 \\ b\mapsto t\end{smallmatrix}}"] & & 1-x-x^2+x^3 = (1-x)^2\cdot(1+x) \\
(1+3y+2y^2)+(2+3y+y^2)t \arrow[urr, mapsto, "\boxed{\begin{smallmatrix} y\mapsto -x \\ t\mapsto x\end{smallmatrix}}"'] & &
\end{tikzcd}$$

Extended ab-index : in $a=1$

The COARSE FLAG POLYNOMIAL of a matroid $\mathcal{M}$ is

$$W_{\mathcal{M}}(y,t) = (1-t)^{\mathrm{rk}(\mathcal{M})} \cdot \mathrm{cfHP}_{\mathcal{M}}(y,t) = \mathrm{ex}\,\widetilde{\Psi}_{\mathcal{L}(\mathcal{M})}(y;1,t)$$

Thm: [Maglione-Voll]

If $\mathcal{A}$ a Coxeter arrangement w/o E_8-factor of rank n, then

$$W_{\mathcal{A}}(1,t) = \#\{\text{regions of } \mathcal{A}\} \cdot \underbrace{\sum_{\sigma \in \mathfrak{S}_n} t^{\mathrm{des}(\sigma)}}_{\substack{\text{Eulerian polynomial,} \\ \text{real-rooted}}}$$

? [Kühne-Maglione]

Is $W_{\mathcal{M}}(1,t)$ real-rooted for all matroids $\mathcal{M}$?

Chow polynomials & Friends

extended ab-index
$\mathrm{ex}\Psi_P(y; a, b)$

$\mathrm{ex}\Psi_P(y;a,b) \xrightarrow[(y;a,b)\mapsto(-x,1,x)]{P\ \mathcal{R}\text{-labeled}}$ Poset Chow polynomial

$\mathrm{ex}\Psi_P(y;a,b) \xrightarrow{P=\mathcal{L}(\mathcal{M})}$ coarse flag polynomial $\mathcal{N}_{\mathcal{M}}(y,t)$

coarse flag polynomial
$\mathcal{N}_{\mathcal{M}}(y,t)$

$\mathcal{N}_{\mathcal{M}}(y,t) \xrightarrow[t\mapsto x]{y\mapsto -x}$ Matroid Chow polynomial

Poset
Chow polynomial

Poset Chow polynomial $\xrightarrow{P=\mathcal{L}(\mathcal{M})}$ Matroid Chow polynomial

$\boxed{\text{Matroid Chow polynomial}}$

κ-Chow function $\xrightarrow{\kappa=\chi}$ Poset Chow polynomial

κ-Chow
function

h-polynomial
of order complex

KL polynomials
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groups

Hilbert-series
of Chow rings
$A(\mathcal{L}, \mathcal{G})$

Chow polynomials & Friends

$\boxed{?}$ Real-rooted?

Stump, 2024

extended ab-index

$ex\Psi_P(y;a,b)$

$P=\mathcal{L}(\mathcal{M})$

coarse flag polynomial

$N_{\mathcal{M}}(y,t)$

Kühne-Maglione, 2023

P R-labeled

$(y;a,b) \mapsto (-x,1,x)$

$y \mapsto -x$

$t \mapsto x$

Poset Chow polynomial

Ferroni-Matherne-Vecchi, 2024

$P=\mathcal{L}(\mathcal{M})$

Matroid Chow polynomial

Ferroni-Schröter, 2023

$K=\chi$

K-Chow function

"

h-polynomial of order complex

KL polynomials of Coxeter groups

Chain polynomial

Athanasiadis-Kalampogia-Evangelinou, 2023

Hilbert-series of Chow rings

$A(\mathcal{L},\mathcal{G})$