Yang-Baxter elements, Jack polynomials, and beyond

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More interesting: Yang-Baxter elements. Appear in

- statistical mechanics,
- braid theory,
- representation theory,
- ...

Yang-Baxter elements are connected to the braid relation.
Indeed, find the conditions on the unknowns to satisfy

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We get six (redundant) equations that simplify into

$$b = b', \qquad a = c', \qquad c = a',$$

and $b = a + c$.

Let us define

$$Y_{s_i}(u,v) = Y_i(u,v) = 1 + (u-v)s_i. \tag{2}$$

Then

$$Y_1(u,v)\,Y_2(u,w)\,Y_1(v,w) = Y_2(v,w)\,Y_1(u,w)\,Y_2(u,v). \tag{3}$$

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Now, given a sequence of *spectral parameters* (historical reasons) $\mathbf{u} = (u_1, \ldots, u_n)$, define

$$Y_{\sigma.s_i}(\mathbf{u}) = Y_\sigma(\mathbf{u})\, Y_i(u_{\sigma_i}, u_{\sigma_{i+1}}). \quad (4)$$

Then Y_σ is well-defined for any permutation σ written as a reduced product of elementary transpositions. Moreover, they form a basis of the symmetric group algebra (triangularity on the strong Bruhat order).

Now define the *trace* that sends each permutation σ to $p_{\lambda(\sigma)}$ where λ is its *cycle type*.
For example, our generic Yang-Baxter element is sent to

$$tr\,(Y_{321}(u,v,w)) = (1+(u-v)(v-w))p_{111} + (u-w)(2+(u-v)(v-w))p_{21} + (u-w)^2 p_3. \quad (5)$$

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Ok, not impressive at all.

The recipe is very similar to get J_3 and J_{21}:

$$tr\,(Y_{312}(u,v,w)) = \; p_{111} + \; (u+v-2w)p_{21} + \; (u-w)(v-w)p_3,$$

which gives $J_3 = p_{111} + 3a\,p_{21} + 2a^2 p_3$ with $u=0$, $v=-a$, $w=-2a$.

And

$$tr\,(Y_{231}(u,v,w)) = \; p_{111} + \; (2u-v-w)p_{21} + \; (u-v)(u-w)p_3,$$

which gives $J_{21} = p_{111} + (a-1)\,p_{21} - a\,p_3$ with $u=0$, $v=1$, $w=-a$.

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Still not very impressive.

We need two informations to compute a Yang-Baxter element: a permutation and a sequence of spectral parameters.

Given a partition λ, fill it with polynomials in a following this simple rule: the cell (r, c) is filled with $(c - r.a)$.
For example,

$-3a$	$1-3a$			
$-2a$	$1-2a$	$2-2a$	$3-2a$	
$-a$	$1-a$	$2-a$	$3-a$	
0	1	2	3	4

(7)

and the sequence $s(5, 4, 4, 2)$ is

$$0, 1, 2, 3, 4, -a, 1-a, 2-a, 3-a, -2a, 1-2a, 2-2a, 3-2a, -3a, 1-3a.$$

Now, about the permutation, write $1, \ldots, n$ in rows from bottom to top and first read from right to left the cells with no values above them then read the remaining values by rows from right to left and from bottom to top.
For example,

14	15			
10	11	12	13	
6	7	8	9	
1	2	3	4	5

(8)

and the permutation $p(5, 4, 4, 2)$ is

$$5, 13, 12, 15, 14, 4, 3, 2, 1, 9, 8, 7, 6, 11, 10.$$

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Other values of p and s also give the Jack polynomials.

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- At the same time, Nakayashiki and Yamada computed energy (Kostka polynomials) in crystal graphs (please ask Anne) and found the same statistics on t on HL pols.
- They found it using the trace of the "combinatorial" R-matrix.
- Why not use the trace of the general R-matrix (that is related to Yang-Baxter elements)?

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- Attempts were made at the time but with wrong assumptions.

The Hecke algebra $H_n(q)$ is the algebra generated by the T_i (with $i \in [1, n-1]$) satisfying

- $T_i T_j = T_j T_i \qquad$ if $|i - j| \geq 2$,
- $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$,
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- Tried to connect it with the R-matrix, hence to traces in the Hecke algebra,
- Related traces with chromatic polynomials thanks to an old conjecture of Haiman on Kazhdan-Lusztig elements (intervals in the Bruhat order),
- Finally ended up computing *all* generic Yang-Baxter elements since the R-matrix connects to Y-B and Lascoux had an algorithm to factorize some intervals in the Bruhat order Y-B-like.

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Of course, Conjecture 2 was made before Conjecture 1.

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Conjecture 3: These multi-t Macdonald have a positive expansion of Schur functions. Moreover, the coefficients of the hook partitions are the elementary symmetric functions of the $q^c t_r$.

Let us look at one coefficient of $\tilde{H}_{321}$:

$$\widetilde{H}_{321} = s_6 + (q^2 + qt_1 + q + t_1 + t_2)s_{51} + \dots$$
$$+ (q^3 t_1^2 + q^4 + 3q^3 t_1 + 2q^2 t_1^2 + q^2 t_1 t_2 + q^2 t_1 + 2q^2 t_2 + 3qt_1 t_2$$
$$+ t_1^2 t_2 + qt_2)s_{321} + \dots$$
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The product of all monomials in s_{321} is $q^{32}t_1^{16}t_2^8 = (q^4t_1^2t_2)^8$.

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The conjecture is true for usual Macdonalds on hook partitions thanks to the remark above.

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Are Macdonald polynomials part of a much larger combinatorial family completely unrelated with the classical ways of thinking about them?