The role of affine Weyl groups in some problems of combinatorics and representation theory

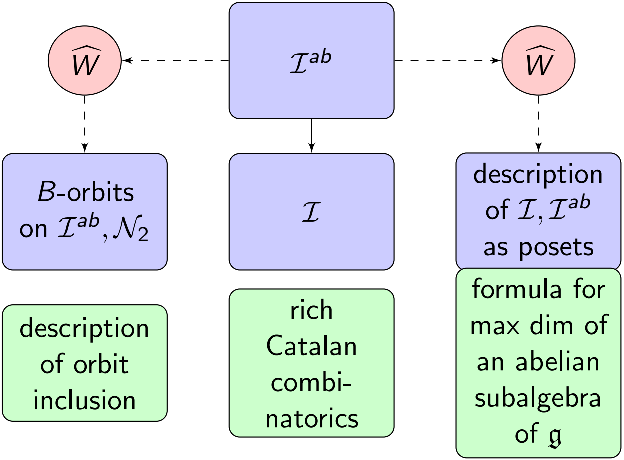
Paolo Papi

Sapienza Università di Roma

Notation

G simple algebraic group over $\mathbb{C}$
$B \subset G$ Borel subgroup, $T \subset B$ maximal torus
$\Delta \supset \Delta^+ \supset \Pi$ roots, positive roots, simple roots
$\mathfrak{g} = Lie\, G, \mathfrak{b} = Lie\, B, \mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$
W Weyl group of $\mathfrak{g}$
$\mathcal{N}$ nilcone of $\mathfrak{g}$
$e \in \mathcal{N}; ht(e) = \max\{n \in \mathbb{N} \mid ad(e)^n \neq 0\}$
$\mathcal{N}_2 = \{e \in \mathcal{N} \mid ht(e) \leqslant 2\}$
$\mathcal{I}$ set of B-stable ideals of $\mathfrak{b}$ contained in $\mathfrak{n}$
$\mathcal{I}^{ab} \subset \mathcal{I}$ abelian ideals of $\mathfrak{b}$
$\widehat{W}$ affine Weyl group of $\mathfrak{g}$

Overview I

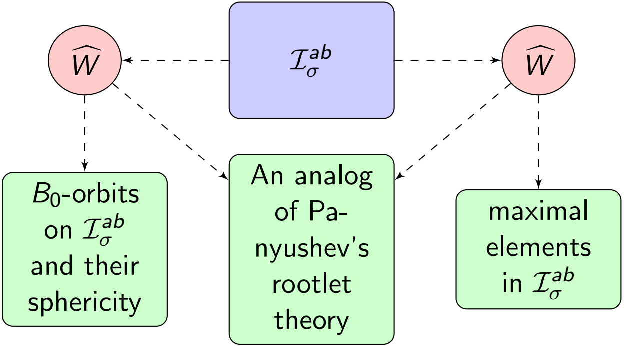


$\widehat{W}$
$\mathcal{I}^{ab}$
$\widehat{W}$
B-orbits on $\mathcal{I}^{ab}, \mathcal{N}_2$
$\mathcal{I}$
description of $\mathcal{I}, \mathcal{I}^{ab}$ as posets
description of orbit inclusion
rich Catalan combi- natorics
formula for max dim of an abelian subalgebra of $\mathfrak{g}$

Overview II

Basic idea for a generalization:

Replace $\mathfrak{g}$ with an infinitesimal symmetric space $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and $\mathcal{I}^{ab}$ with the set $\mathcal{I}^{ab}_\sigma$ of commutative subalgebras of $\mathfrak{p}$ stable w.r.t. to a Borel subalgebra of $\mathfrak{k}$.



Abelian ideals of Borel subalgebras

$\mathcal{I}^{ab}$ abelian ideals of $\mathfrak{b}$.

$\mathfrak{t} \subset \mathfrak{b}$ Cartan subalgebra

$\Delta \supset \Delta^+ \supset \Pi$, $(\mathfrak{g}, \mathfrak{t})$-root system, positive system, simple roots

Abelian ideals of Borel subalgebras

$\mathcal{I}^{ab}$ abelian ideals of $\mathfrak{b}$.
$\mathfrak{t} \subset \mathfrak{b}$ Cartan subalgebra
$\Delta \supset \Delta^+ \supset \Pi$, $(\mathfrak{g}, \mathfrak{t})$-root system, positive system, simple roots

$$\mathcal{I} \ni \mathfrak{i} = \bigoplus_{\alpha \in \Phi(\mathfrak{i})} \mathfrak{g}_\alpha,$$

$\Phi(\mathfrak{i}) \subset \Delta^+$ "abelian" dual order ideal of the root poset

Abelian ideals of Borel subalgebras

$\mathcal{I}^{ab}$ abelian ideals of $\mathfrak{b}$.
$\mathfrak{t} \subset \mathfrak{b}$ Cartan subalgebra
$\Delta \supset \Delta^+ \supset \Pi$, $(\mathfrak{g}, \mathfrak{t})$-root system, positive system, simple roots

$$\mathcal{I} \ni \mathfrak{i} = \bigoplus_{\alpha \in \Phi(\mathfrak{i})} \mathfrak{g}_\alpha,$$

$\Phi(\mathfrak{i}) \subset \Delta^+$ "abelian" dual order ideal of the root poset

Example: $sl(3)$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & * & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix},$$

Abelian ideals of Borel subalgebras

$\mathcal{I}^{ab}$ abelian ideals of $\mathfrak{b}$.
$\mathfrak{t} \subset \mathfrak{b}$ Cartan subalgebra
$\Delta \supset \Delta^+ \supset \Pi$, $(\mathfrak{g}, \mathfrak{t})$-root system, positive system, simple roots

$$\mathcal{I} \ni \mathfrak{i} = \bigoplus_{\alpha \in \Phi(\mathfrak{i})} \mathfrak{g}_\alpha,$$

$\Phi(\mathfrak{i}) \subset \Delta^+$ "abelian" dual order ideal of the root poset

Example: $sl(n)$

Combinatorially

$\mathcal{I}$	$\leftrightarrow$	subdiagrams of the staircase shape
$\mathcal{I}^{ab}$	$\leftrightarrow$	subdiagrams as above with hook length of box $(1,1) \leqslant n$

Peterson's theory

$\widehat{\Delta}(=\widehat{\Delta}^+_{re})$ affine root system, δ fundamental imaginary root, $\widehat{W}$ affine Weyl group

Peterson's theory

$\widehat{\Delta}(=\widehat{\Delta}^+_{re})$ affine root system, δ fundamental imaginary root, $\widehat{W}$ affine Weyl group Recall that

$$\widehat{\Delta}^+ = (\Delta^+ + \mathbb{Z}_{\geqslant 0}\delta) \cup (-\Delta^+ + \mathbb{N}\delta)$$

Peterson's theory

Fact (Peterson, advertised by Kostant, 1999)

If $\mathfrak{i} \in \mathcal{I}^{ab}$, then $-\Phi(\mathfrak{i}) + \delta \subset \widehat{\Delta}^+$ is biconvex in $\widehat{\Delta}^+$, hence there exists a unique $w_{\mathfrak{i}} \in \widehat{W}$ such that

$$N(w) := \{\alpha \in \widehat{\Delta}^+ \mid w^{-1}(\alpha) < 0\} = -\Phi(\mathfrak{i}) + \delta.$$

Proposition (Cellini-P.)

Let A be the fundamental alcove. Given $w \in \widehat{W}$, there exists $\mathfrak{i} \in \mathcal{I}^{ab}$ such that $w = w_{\mathfrak{i}}$ iff $wA \subset 2A$.

Peterson's theory

Fact (Peterson, advertised by Kostant, 1999)

If $\mathfrak{i} \in \mathcal{I}^{ab}$, then $-\Phi(\mathfrak{i}) + \delta \subset \widehat{\Delta}^+$ is biconvex in $\widehat{\Delta}^+$, hence there exists a unique $w_\mathfrak{i} \in \widehat{W}$ such that

$$N(w) := \{\alpha \in \widehat{\Delta}^+ \mid w^{-1}(\alpha) < 0\} = -\Phi(\mathfrak{i}) + \delta.$$

Proposition (Cellini-P.)

Let A be the fundamental alcove. Given $w \in \widehat{W}$, there exists $\mathfrak{i} \in \mathcal{I}^{ab}$ such that $w = w_\mathfrak{i}$ iff $wA \subset 2A$.

Remark

Notice that the above proposition immediately implies the so-called *Peterson's abelian ideal theorem*: $|\mathcal{I}^{ab}| = 2^{\operatorname{rk}\mathfrak{g}}$.

Peterson's theory

Fact (Peterson, advertised by Kostant, 1999)

If $\mathfrak{i} \in \mathcal{I}^{ab}$, then $-\Phi(\mathfrak{i}) + \delta \subset \widehat{\Delta}^+$ is biconvex in $\widehat{\Delta}^+$, hence there exists a unique $w_\mathfrak{i} \in \widehat{W}$ such that

$$N(w) := \{\alpha \in \widehat{\Delta}^+ \mid w^{-1}(\alpha) < 0\} = -\Phi(\mathfrak{i}) + \delta.$$

Proposition (Cellini-P.)

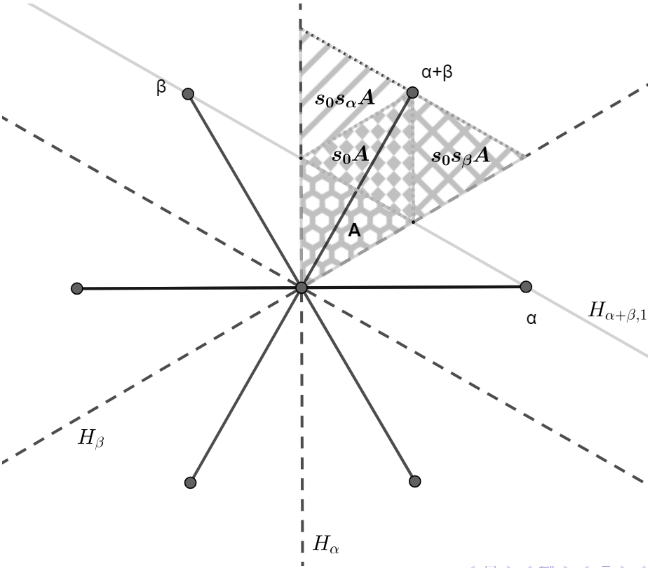
Let A be the fundamental alcove. Given $w \in \widehat{W}$, there exists $\mathfrak{i} \in \mathcal{I}^{ab}$ such that $w = w_\mathfrak{i}$ iff $wA \subset 2A$.

Remark

Notice that the above proposition immediately implies the so-called *Peterson's abelian ideal theorem*: $|\mathcal{I}^{ab}| = 2^{\mathrm{rk}\,\mathfrak{g}}$.

Notation: We define the *minuscule* elements as $\mathcal{W}^{ab} = \{w_\mathfrak{i} \mid \mathfrak{i} \in \mathcal{I}^{ab}\}$.

An illustration



Description of the B-orbits

If $S \subset \Delta$ is a set of orthogonal roots define

$$e_S = \sum_{\alpha \in S} e_\alpha,$$

for a fixed choice of root vectors e_α.

Description of the B-orbits

If $S \subset \Delta$ is a set of orthogonal roots define

$$e_S = \sum_{\alpha \in S} e_\alpha,$$

for a fixed choice of root vectors e_α.

Theorem (Panyushev)

Let $\mathfrak{i} \in \mathcal{I}^{ab}$. Then the set of B-orbits in $\mathfrak{i}$ is in bijection with set of orthogonal subsets of $\Phi(\mathfrak{i})$ via the map $S \mapsto Be_S$.

Description of the B-orbits

Example

$$\mathfrak{g} = sl(4), \mathfrak{i} = \mathfrak{g}_\theta \oplus \mathfrak{g}_{\alpha_1+\alpha_2} \oplus \mathfrak{g}_{\alpha_2+\alpha_3} \oplus \mathfrak{g}_{\alpha_2} = \begin{pmatrix} 0 & 0 & \alpha_1+\alpha_2 & \theta \\ 0 & 0 & \alpha_2 & \alpha_2+\alpha_3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

There are 7 orbits, indexed by

$$S_1 = \emptyset, S_2 = \{\theta\}, S_3 = \{\alpha_1+\alpha_2\}, S_4 = \{\alpha_2+\alpha_3\}, S_5 = \{\alpha_2\},$$

$$S_6 = \{\alpha_1+\alpha_2, \alpha_2+\alpha_3\}, S_7 = \{\theta, \alpha_2\}$$

Main Theorem

If $S \subset \widehat{\Delta}$ is a finite set of orthogonal real roots we define

$$\sigma_S = \prod_{\alpha \in S} s_\alpha.$$

Main Theorem

If $S \subset \widehat{\Delta}$ is a finite set of orthogonal real roots we define

$$\sigma_S = \prod_{\alpha \in S} s_\alpha .$$

Theorem (Gandini-Maffei-Möseneder-P.)

Let $\mathfrak{a}$ be an abelian ideal of $\mathfrak{b}$ and let $S, T \subset \Phi(\mathfrak{a})$ be two sets of orthogonal roots. Define $\widehat{S} = S - \delta$ and $\widehat{T} = T - \delta$. Then

$$\dim Be_S = \frac{\ell(\sigma_{\widehat{S}}) + \mathrm{card}(S)}{2}$$

and

$$e_S \in \overline{Be_T} \quad \textit{if and only if} \quad \sigma_{\widehat{S}} \leqslant \sigma_{\widehat{T}}.$$

Illustration of the theorem in the above example

$$S_1=\emptyset, S_2=\{\theta\}, S_3=\{\alpha_1+\alpha_2\}, S_4=\{\alpha_2+\alpha_3\}, S_5=\{\alpha_2\},$$
$$S_6=\{\alpha_1+\alpha_2,\alpha_2+\alpha_3\}, S_7=\{\theta,\alpha_2\}$$

$\sigma_{\widehat{S}}$	$\dim Be_S$
$\sigma_{\widehat{S}_1}=e$	0
$\sigma_{\widehat{S}_2}=s_{-\delta+\theta}=s_0$	1
$\sigma_{\widehat{S}_3}=s_{-\delta+(\alpha_1+\alpha_2)}=s_{\alpha_0+\alpha_3}=s_0s_3s_0$	2
$\sigma_{\widehat{S}_4}=s_{-\delta+(\alpha_2+\alpha_3)}=s_{\alpha_0+\alpha_1}=s_0s_1s_0$	2
$\sigma_{\widehat{S}_5}=s_{-\delta+\alpha_2}=s_{\alpha_0+\alpha_1+\alpha_3}=s_1s_0s_3s_0s_1$	3
$\sigma_{\widehat{S}_6}=s_{-\delta+(\alpha_2+\alpha_3)}s_{-\delta+(\alpha_1+\alpha_2)}=s_0s_3s_1s_0$	3
$\sigma_{\widehat{S}_7}=s_{-\delta+\theta}s_{-\delta+\alpha_2}=s_0s_1s_0s_3s_0s_1$	4

Some developments (Gandini-Möseneder-P.)

Theorem

Let $R, S \subset \Delta$ be strongly orthogonal with $\mathrm{ht}(e_R) = \mathrm{ht}(e_S) = 2$, then $Be_R \subset \overline{Be_S}$ if and only if $\sigma_{\widehat{R}} \leqslant \sigma_{\widehat{S}}$. Moreover, we have

$$\dim(Be_S) = \frac{\ell(\sigma_{\widehat{S}}) + |S|}{2}.$$

Some developments (Gandini-Möseneder-P.)

Theorem

Let $R, S \subset \Delta$ be strongly orthogonal with $\mathrm{ht}(e_R) = \mathrm{ht}(e_S) = 2$, then $Be_R \subset \overline{Be_S}$ if and only if $\sigma_{\widehat{R}} \leqslant \sigma_{\widehat{S}}$. Moreover, we have

$$\dim(Be_S) = \frac{\ell(\sigma_{\widehat{S}}) + |S|}{2}.$$

Suppose now that $\mathcal{O} \subset \mathcal{N}_2$; to $\mathcal{O}$ we can associate an abelian ideal $\mathfrak{a}$ and a parabolic P. Set $\widetilde{\mathcal{O}} = G \times_P \mathfrak{a}$. In this case the orbit structure of $\widetilde{\mathcal{O}}$ largely reduces to that of the abelian ideal $\mathfrak{a} \subset \mathfrak{b}$.

Theorem

Suppose that $\mathcal{O} \subset \mathcal{N}_2$ and let (v, R), (w, S) be in $W^P \times \mathrm{Ort}(\Phi(\mathfrak{a}) \cong B/\tilde{\mathcal{O}}$. Then $B[v, e_R] \subset \overline{B[w, e_S]}$ if and only if $v \leqslant w$ and $\sigma_{v(\widehat{R})} \leqslant \sigma_{w(\widehat{S})}$.

A few words on $\mathcal{I}$

1. B-stable ideals are again encoded by $\widehat{W}$ using same trick as for $\mathcal{I}^{ab}$; the correct set of inversion to be used is this time

$$L_{\mathfrak{i}} = \bigcup_{k\geqslant 1}(-\Phi^k(\mathfrak{i}) + k\delta).$$

2. Enumeration is reduced to a lattice point enumeration by using the decomposition $\widehat{W} = W \ltimes Q^\vee$; the result is the generalized Catalan number

$$\prod_{i=1}^{\dim \mathfrak{t}} \frac{(h + e_i + 1)}{|W|}$$

$e_i\ \ldots, e_{\dim \mathfrak{t}}$ being the exponents of W.

3. B-stable ideals are related to the Shi arrangement, which is given by the hyperplanes $(\alpha, x) \in \{0, \pm 1\}, \alpha \in \Delta^+$. The regions contained in the fundamental chamber have a minimal alcove which is index by $w_{\mathfrak{i}}$ when $\mathfrak{i}$ ranges ove $\mathcal{I}$.

A few words on $\mathcal{I}$

1. B-stable ideals also play a role in the theory of nilpotent orbits: $G\mathfrak{i}$ turns out to be the closure of a single nilpotent orbit $\mathcal{O}_{\mathfrak{i}}$. Fenn and Sommers describe the equivalence relation on $\mathcal{I}$ under which $\mathcal{O}_{\mathfrak{i}} = \mathcal{O}_{\mathfrak{j}}$. In type A, a single *move* suffices. All is inspired by the following result by De Concini-Lusztig-Procesi:

Proposition

If $\mathfrak{i} \subset \mathfrak{j}$ are B-stable ideals, $\mathfrak{j} = \mathfrak{i} + \mathfrak{g}_\beta$ and $\mathfrak{j}$ is stable under a minimal parabolic P_α such that $\beta(\alpha^\vee) = -1$, then $\mathcal{O}_{\mathfrak{i}} = \mathcal{O}_{\mathfrak{j}}$.

A short historical perspective on $\mathcal{I}^{ab}$

- *Schur 1905*: There exist at most $\lfloor \frac{N^2}{4} \rfloor + 1$ linearly independent commuting matrices in $gl(N)$. See also Mirzakhani 1998 AMM

A short historical perspective on $\mathcal{I}^{ab}$

- *Schur 1905*: There exist at most $\lfloor \frac{N^2}{4} \rfloor + 1$ linearly independent commuting matrices in $gl(N)$. See also Mirzakhani 1998 AMM
- *Malcev 1945*: Maximal dimension of commutative subalgebras of any simple Lie algebra (case by case calculations).

A short historical perspective on $\mathcal{I}^{ab}$

- *Schur 1905*: There exist at most $\lfloor \frac{N^2}{4} \rfloor + 1$ linearly independent commuting matrices in $gl(N)$. See also Mirzakhani 1998 AMM
- *Malcev 1945*: Maximal dimension of commutative subalgebras of any simple Lie algebra (case by case calculations).
- *Kostant 1965*: Link between the eigenvalues of a Casimir operator and a remarkable submodule of $\Lambda\mathfrak{g}$, related to the commutative subalgebras of $\mathfrak{g}$.

A short historical perspective on $\mathcal{I}^{ab}$

- *Schur 1905*: There exist at most $\lfloor \frac{N^2}{4} \rfloor + 1$ linearly independent commuting matrices in $gl(N)$. See also Mirzakhani 1998 AMM
- *Malcev 1945*: Maximal dimension of commutative subalgebras of any simple Lie algebra (case by case calculations).
- *Kostant 1965*: Link between the eigenvalues of a Casimir operator and a remarkable submodule of $\Lambda\mathfrak{g}$, related to the commutative subalgebras of $\mathfrak{g}$.
- *Peterson 1999*: The abelian ideals of a Borel subalgebra of $\mathfrak{g}$ are $2^{rank(\mathfrak{g})}$.

A short historical perspective on $\mathcal{I}^{ab}$

- *Schur 1905*: There exist at most $\lfloor \frac{N^2}{4} \rfloor + 1$ linearly independent commuting matrices in $gl(N)$. See also Mirzakhani 1998 AMM
- *Malcev 1945*: Maximal dimension of commutative subalgebras of any simple Lie algebra (case by case calculations).
- *Kostant 1965*: Link between the eigenvalues of a Casimir operator and a remarkable submodule of $\Lambda\mathfrak{g}$, related to the commutative subalgebras of $\mathfrak{g}$.
- *Peterson 1999*: The abelian ideals of a Borel subalgebra of $\mathfrak{g}$ are $2^{rank(\mathfrak{g})}$.
- *Panyushev 2003*: Natural bijection between maximal abelian ideals and long simple roots; rootlets theory.

A short historical perspective on $\mathcal{I}^{ab}$

- *Schur 1905*: There exist at most $\lfloor \frac{N^2}{4} \rfloor + 1$ linearly independent commuting matrices in $gl(N)$. See also Mirzakhani 1998 AMM
- *Malcev 1945*: Maximal dimension of commutative subalgebras of any simple Lie algebra (case by case calculations).
- *Kostant 1965*: Link between the eigenvalues of a Casimir operator and a remarkable submodule of $\Lambda\mathfrak{g}$, related to the commutative subalgebras of $\mathfrak{g}$.
- *Peterson 1999*: The abelian ideals of a Borel subalgebra of $\mathfrak{g}$ are $2^{rank(\mathfrak{g})}$.
- *Panyushev 2003*: Natural bijection between maximal abelian ideals and long simple roots; rootlets theory.
- *Suter 2004*: conceptual approach to Malcev's results.

Generalization to the graded case

Abelian Case

- Kostant's Theorem
- Peterson's Theorem

- Suter's formula on the dimension of maximal elements
- Panyushev's rootlets theory

Generalization to the graded case

Abelian Case

- Kostant's Theorem
- Peterson's Theorem
- Suter's formula on the dimension of maximal elements
- Panyushev's rootlets theory

$\mathbb{Z}_2$-graded case

- Panyushev
- Cellini-Möseneder-P.
- Cellini-Möseneder-P.-Pasquali
- Stara (2021) for non hermitian involutions
- Möseneder Frajria and P. : hermitian symmetric case, in progress)

Panyushev bijection

Fact

If $w \in \mathcal{W}^{ab}$, then $w^{-1}(-\theta + 2\delta) \in \Delta_\ell^+$. Set for $\beta \in \Delta_\ell^+$

$$\mathcal{I}_\beta = \{\mathfrak{i} \mid w_\mathfrak{i}^{-1}(-\theta + 2\delta) = \beta\}$$

Panyushev bijection

Fact

If $w \in \mathcal{W}^{ab}$, then $w^{-1}(-\theta + 2\delta) \in \Delta_\ell^+$. Set for $\beta \in \Delta_\ell^+$

$$\mathcal{I}_\beta = \{\mathfrak{i} \mid w_\mathfrak{i}^{-1}(-\theta + 2\delta) = \beta\}$$

Proposition (Panyushev, Suter)

Set $\mathcal{I}_{ab} = \mathcal{I}^{ab} \setminus \{0\}$. *Then*

$$\mathcal{I}_{ab} = \bigsqcup_{\alpha \in \Delta_\ell^+} \mathcal{I}_\alpha$$

Panyushev bijection

Fact

If $w \in \mathcal{W}^{ab}$, then $w^{-1}(-\theta + 2\delta) \in \Delta_\ell^+$. Set for $\beta \in \Delta_\ell^+$

$$\mathcal{I}_\beta = \{\mathfrak{i} \mid w_\mathfrak{i}^{-1}(-\theta + 2\delta) = \beta\}$$

Proposition (Panyushev, Suter)

Moreover

$$\mathcal{I}_\alpha \cong \widehat{W}_\alpha / W_\alpha$$

where

$$\widehat{W}_\alpha = \langle s_\beta \mid \beta \in \widehat{\Pi}, \beta \perp \alpha \rangle, W_\alpha = \langle s_\beta \mid \beta \in \Pi, \beta \perp \alpha \rangle$$

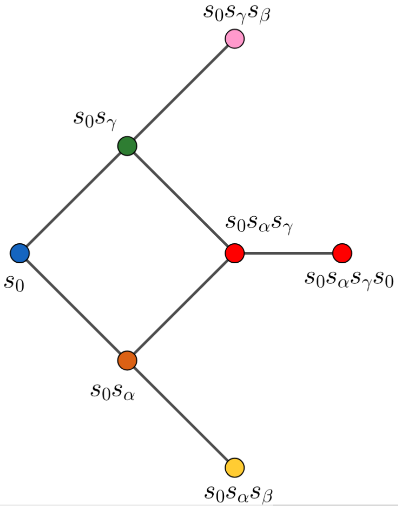
In particular, $\mathcal{I}_\alpha$ has minimum and maximum. Moreover, the map

$$\beta \mapsto \max \mathcal{I}_\beta$$

sets up a bijection $\{$long simple roots$\} \leftrightarrow \{$maximal abelian ideals$\}$.

An illustration: *sl*(4), with Dynkin diagram $\alpha - - - \beta - - - \gamma$

- $\boldsymbol{\alpha+\beta+\gamma}$
- $\boldsymbol{\alpha+\beta}$
- $\boldsymbol{\beta+\gamma}$
- $\boldsymbol{\alpha}$
- $\boldsymbol{\beta}$



Setting for the graded case

Let

- $\mathfrak{g}$ be a simple finite dimensional complex Lie algebra of rank n,
- σ be an involution of $\mathfrak{g}$, defined by Kac parameters $(s_0, \ldots, s_n; k)$.
- $\mathfrak{g} = \mathfrak{g}^{\bar{0}} \oplus \mathfrak{g}^{\bar{1}}$ eigenspace decomposition
- let $\mathfrak{b}^{\bar{0}}$ be a Borel subalgebra of $\mathfrak{g}^{\bar{0}}$.

Goal

Study the poset of abelian subalgebras of $\mathfrak{g}^{\bar{1}}$ which are $\mathfrak{b}^{\bar{0}}$-stable.

Setting for the graded case

Let $(s; k)$ be such that $k\Big(\sum_{i=0}^n a_i s_i\Big) = 2$. For $\alpha = \sum_{i=0}^n c_i\alpha_i \in \widehat{\Delta}$, define the *$\sigma$-height*

$$h_\sigma(\alpha) = \sum_{i=0}^n c_i s_i$$

Definition

We call **$\boldsymbol{\sigma}$-minuscule** the elements in $\widehat{W}$ such that $N(w)$ consists of roots of σ-height 1. $\widehat{W}_\sigma^{ab}$ is called the set of σ-minuscule elements, and we regard it as a poset under the weak Bruhat order.

Setting for the graded case

The following theorem translates the problem of studying the poset of abelian subalgebras contained in $\mathfrak{g}^{\bar{1}}$ and $\mathfrak{b}^{\bar{0}}$-stable into a combinatorial problem:

Theorem (Cellini - Möseneder Frajria - P.)

Let $w \in \widehat{W}_\sigma^{ab}$ with $N(w) = \{\alpha_1, \dots, \alpha_m\}$. Then the map $\widehat{W}_\sigma^{ab} \to \mathcal{I}_{ab}^\sigma$

$$w \mapsto \bigoplus_{i=1}^{m} \mathfrak{g}^{\bar{1}}_{-\bar{\alpha}_i}$$

is a poset isomorphism. Moreover $\widehat{W}_\sigma^{ab}$ can be described as a polytope in affine space defined by explicit equations depending only on the Kac parameters of σ.

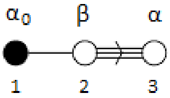
Example

Consider the Lie algebra $G_2 = \langle \alpha, \beta \rangle$, with β long. Fix the $\mathbb{Z}_2$-grading

$$\mathfrak{g}^{\bar{0}} = L_\alpha \oplus L_{-\alpha} \oplus L_{3\alpha+2\beta} \oplus L_{-3\alpha-2\beta} \oplus \mathfrak{h}$$

$$\mathfrak{g}^{\bar{1}} = L_\beta \oplus L_{-\beta} \oplus L_{\beta+\alpha} \oplus L_{-\beta-\alpha} \oplus L_{\beta+2\alpha} \oplus L_{-\beta-2\alpha} \oplus L_{\beta+3\alpha} \oplus L_{-\beta-3\alpha}.$$

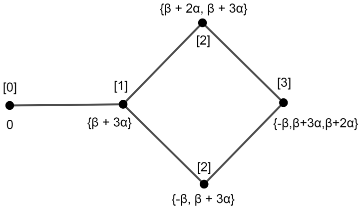
The corresponding diagram is $G_2^{(1)}$



$k = 1$ and the $(n+1)$-tuple is $s = (0, 1, 0)$, so $\Pi_0 = \{\alpha_0, \alpha\}$ and $\Pi_1 = \{\beta\}$.

Example

This is the poset of abelian subalgebras contained in $\mathfrak{g}^{\bar{1}}$ and $\mathfrak{b}^{\bar{0}}$-stable.

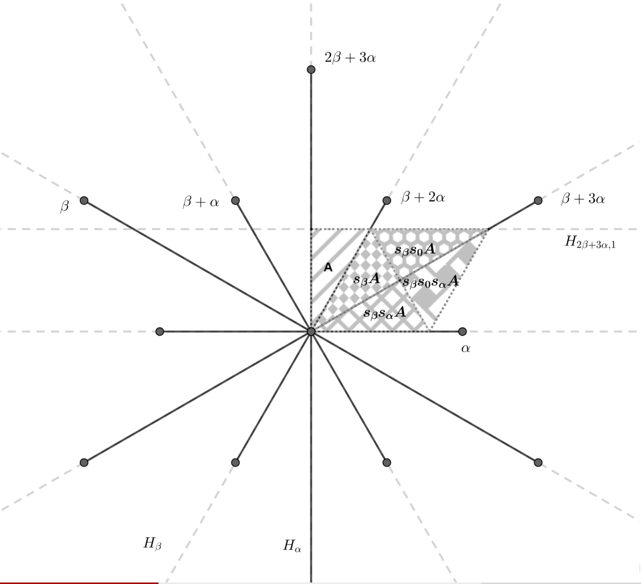


$$\widehat{\Delta}_1 = \{\beta, \beta+\alpha, \beta+2\alpha, \beta+3\alpha, \delta-3\alpha-\beta, \delta-2\alpha-\beta, \delta-\alpha-\beta, \delta-\beta\}.$$

Its non-empty biconvex subsets are
$\{\beta\}, \{\beta, \beta+\alpha\}, \{\beta, \delta-3\alpha-\beta\}, \{\beta, \delta-3\alpha-\beta, \beta+\alpha\} \implies$

$$\widehat{W}_\sigma^{ab} = \{1, s_\beta, s_\beta s_\alpha, s_\beta s_0, s_\beta s_0 s_\alpha\}.$$

Example



$2\beta+3\alpha$
β
$\beta+\alpha$
$\beta+2\alpha$
$\beta+3\alpha$
$H_{2\beta+3\alpha,1}$
A
$s_\beta s_0 A$
$s_\beta A$
$s_\beta s_0 s_\alpha A$
$s_\beta s_\alpha A$
α
H_β
H_α

Structure theory: maximal elements

Proposition (Cellini-Möseneder-P.-Pasquali)

Let $\mathcal{M}_\sigma$ be the set of non central walls of the polytope D_σ. If $w \in \widehat{W}_\sigma^{ab}$ is maximal, then there exist $\alpha \in \widehat{\Pi}$ and $\mu \in \mathcal{M}_\sigma$ such that $w(\alpha) = \mu$.

In order to study the maximal elements in $\widehat{W}_\sigma^{ab}$, the main point was to study the following subposets: given $\alpha \in \widehat{\Pi}$ and $\mu \in \mathcal{M}_\sigma$, define

$$\mathcal{I}_{\alpha,\mu} = \{w \in \widehat{W}_\sigma^{ab} \mid w(\alpha) = \mu\}.$$

We

- found necessary and sufficient conditions under which the posets $\mathcal{I}_{\alpha,\mu}$ are non empty,
- proved that $\mathcal{I}_{\alpha,\mu}$ have minimum,
- proved that $\mathcal{I}_{\alpha,\mu}$ are isomorphic to the set of minimal right coset representatives for a suitable pair of subgroups of $\widehat{W}$,
- we investigated their intersections.

The poset $\mathcal{I}_{\alpha,\mu}$ and its minimal elements

Given $\alpha \in \widehat{\Pi}$, $\mu \in \mathcal{M}_\sigma$, set

$$\mathcal{I}_{\alpha,\mu} = \{w \in \widehat{W}_\sigma^{ab} \mid w(\alpha) = \mu\}.$$

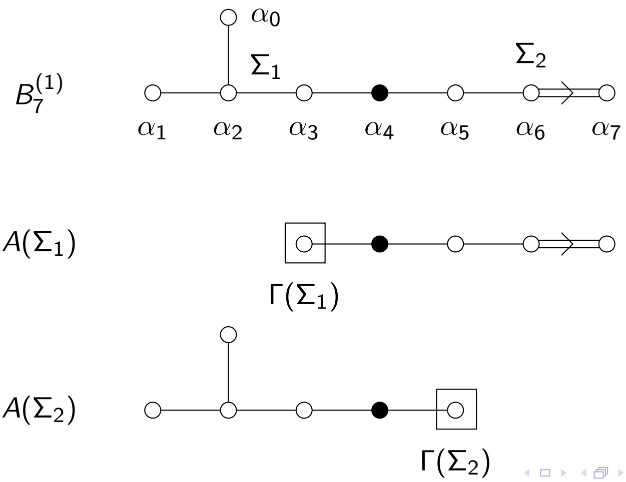
Definition

Let Σ be a connected component of Π_0, and consider the subgraph of $\widehat{\Pi}$ with $\{\alpha \in \widehat{\Pi} \mid (\alpha, \theta_\Sigma) \leqslant 0\}$ as set of vertices. We call $A(\Sigma)$ the union of the connected components of this subgraph which contain at least one root of Π_1. Moreover, we set

$$\Gamma(\Sigma) = A(\Sigma) \cap \Sigma.$$

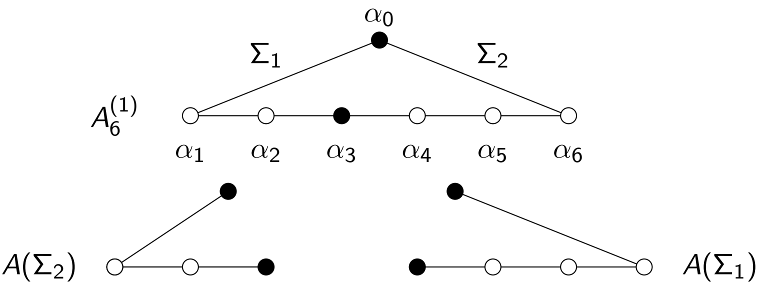
Example: $\widehat{\Delta}$ be of type $B_n^{(1)}$, $\Pi_0 = \Sigma_1 \sqcup \Sigma_2 \cong D_p \times B_{n-p}$.

$A(\Sigma_1) = \{\alpha_{p-1}, \dots, \alpha_n\},\ \Gamma(\Sigma_1) = \{\alpha_{p-1}\}, A(\Sigma_2) = \{\alpha_0, \dots, \alpha_{p+1}\},\ \Gamma(\Sigma_2) = \{\alpha_{p+1}\}$



Example: $A_n^{(1)}, \Pi_1 = \{\alpha_0, \alpha_p\}, \Pi_0 \cong A_{p-1} \times A_{n-p}$

$$A(\Sigma_1) = \Sigma_2 \cup \Pi_1, A(\Sigma_2) = \Sigma_1 \cup \Pi_1,\ \Gamma(\Sigma_i) = \emptyset, i = 1, 2.$$



Set

$$w_{\alpha,\mu} = \begin{cases} w_\alpha & \text{if } \mu = k\delta - \theta_\Sigma,\ \theta_\Sigma \text{ is of type 1, and } \alpha \in A(\Sigma)_{\overline{\ell}} \\ s v_\alpha & \text{if } \mu = k\delta - \theta_\Sigma,\ \theta_\Sigma \text{ is of type 2, and } \alpha \in \Sigma_{\overline{\ell}} \\ s_\beta w_{0,\beta} w_0 & \text{if } \mu = \beta + k\delta,\ \beta \in \Pi_1 \end{cases}$$

and

$$\widehat{\Pi}_\mu = \begin{cases} A(\Sigma)_{\overline{\ell}} & \text{if } \mu = k\delta - \theta_\Sigma \text{ and } \theta_\Sigma \text{ is of type 1} \\ \Sigma_{\overline{\ell}} & \text{if } \mu = k\delta - \theta_\Sigma \text{ and } \theta_\Sigma \text{ is of type 2} \\ \Pi_1 & \text{if } \mu = \beta + k\delta \text{ and } \{\beta\} = \Pi_1 \\ \Pi_1 \setminus \{\beta\} & \text{if } \mu = \beta + k\delta,\ \beta \in \Pi_1, \text{ and } |\Pi_1| = 2 \end{cases}$$

Theorem

Assume $\mu \in \mathcal{M}_\sigma$ and $\alpha \in \widehat{\Pi}$. Then $\mathcal{I}_{\alpha,\mu} \neq \emptyset$ if and only if $\alpha \in \widehat{\Pi}_\mu$. Moreover,

$$w_{\alpha,\mu} = \min \mathcal{I}_{\alpha,\mu}.$$

A rootlet theory for $\mathfrak{b}^{\bar{0}}$-stable abelian subspaces

Given $\alpha \in \widehat{\Delta}^+$, $\mu \in \mathcal{M}_\sigma$, set, extending the previous definition

$$\mathcal{I}_{\alpha,\mu} = \{w \in \widehat{W}_\sigma^{ab} \mid w(\alpha) = \mu\}.$$

Fix $\mu \in \mathcal{M}_\sigma$. Then, clearly

$$\widehat{W}_\sigma^{ab} = \bigsqcup_{\alpha \in \widehat{\Delta}^+} \mathcal{I}_{\alpha,\mu}.$$

The Hermitian symmetric case: non central *ordinary walls*

Now σ is a Hermitian symmetric involution, determined by two simple roots $\{\alpha_p, \alpha_q\}$ having label 1 in $\widehat{\Pi}$. Let Φ denote the set of diagram automorphisms of $\widehat{\Pi}$.

Proposition

Consider an ordinary wall: $\mu = \delta - \theta_\Sigma$. *Then*

$$\mathcal{I}_{\alpha,\mu} \neq \emptyset \iff \alpha \in \langle A(\Sigma)\rangle \cup \bigcup_{\varphi\in\Phi,\varphi(\Sigma)\neq\Sigma} (\langle A(\varphi(\Sigma))\rangle \cap \widehat{\Delta}_1)$$

Examples

1. $\mathfrak{g}$ of type E_7, $\Pi = \{\alpha_7, \alpha_8\}$, $\Sigma = E_6(1,2,3,4,5,6)$, $\delta - \theta_\Sigma = \theta_{A_7(1,3,4,5,6,7,8)}$. In this case Φ consists of the identity an of the simmetry φ which fixes α_2, α_4; in particular, $\varphi(\Sigma) = \Sigma$, hence $\mathcal{I}_{\alpha,\mu} \neq \emptyset \iff \alpha \in A_7(1,3,4,5,6,7,8)$.
2. $\mathfrak{g}$ of type E_6, $\Pi = \{\alpha_1, \alpha_7\}$, $\Sigma = D_5(2,3,4,5,6)$, $\delta - \theta_\Sigma = \theta_{A_5(1,3,4,2,7)}$. In this case Φ consists of the identity and of the simmetries φ_1, φ_2 which map α_1, α_7 to α_1, α_6 and α_1, α_7 to α_6, α_7, respectively. Hence
$$\begin{aligned}\mathcal{I}_{\alpha,\mu} \neq \emptyset \iff \alpha \in & A_5(1,3,4,2,7)\\ & \cup \{\alpha_1+\alpha_3+\alpha_4+\alpha_5, \alpha_1+\alpha_3+\alpha_4+\alpha_5+\alpha_6\}\\ & \cup \{\alpha_2+\alpha_4+\alpha_5+\alpha_7, \alpha_2+\alpha_4+\alpha_5+\alpha_6+\alpha_7\}\end{aligned}$$
3. $\mathfrak{g}$ of type A_6, $\Pi = \{\alpha_4, \alpha_7\}$, $\Sigma = \Sigma_1 \times \Sigma_2 = A_3(1,2,3) \times A_2(5,6)$,
$$\begin{aligned}\mathcal{I}_{\alpha,\delta-\theta_{\Sigma_1}} \neq \emptyset \iff \alpha \in & A_4(4,5,6,7)\\ & \cup \{34, 345, 3456\} \cup \{234, 2345\} \cup \{1234\}\end{aligned}$$

Special walls

Proposition

Consider a special wall $\mu = \delta + \alpha_p$. Then for $\alpha = \sum_{\alpha \in \widehat{\Pi}} c_\alpha(\beta)\alpha \in \widehat{\Delta}_l^+$,

$$\mathcal{I}_{\alpha,\mu} \neq \emptyset \iff \alpha \in \begin{array}{l} \{\beta \mid c_p(\beta) = 1, c_q(\beta) = 0\} \\ \cup \{\delta + \beta \mid c_p(\beta) = 0, c_q(\beta) = 1\} \\ \cup \{2\delta + \alpha_q\}. \end{array}$$

A rootlets theory for $\mathfrak{b}^{\bar{0}}$-stable abelian subspaces

I will illustrate the results of the Ph.D thesis of my former student Federico Stara on the semisimple case, i.e. $\Pi_0 = \{\beta\}$.

Name	Type of wall	Length of β	Type of θ_Σ	$\|\Sigma\|$
a	$k\delta - \theta_\Sigma$	long	1	> 1
b	$k\delta - \theta_\Sigma$	long	1	1
c	$k\delta - \theta_\Sigma$	long	2	1
d	$k\delta + \beta$	long		
e	$k\delta - \theta_\Sigma$	long	2	> 1
f	$k\delta - \theta_\Sigma$	short		
g	$k\delta + \beta$	short		

Main Theorem

Define $\widehat{\Delta}_\mu$ according to the following table.

Case	$\widehat{\Delta}_\mu$
a	$\langle A(\Sigma)\rangle_\ell$
b	$\langle A(\Sigma)\rangle_\ell \cup (\{\delta - \langle A(\Sigma)\rangle_\ell\})$
c	$\widehat{\Delta}^1_{\theta_\Sigma}$ if $\beta \leftrightarrow \theta_\Sigma$ is a double link, $\{\gamma \in (k\delta)^< : \lvert\gamma\rvert = \lvert\theta_\Sigma\rvert, \gamma \neq \theta_\Sigma\}$ oth
d	$\widehat{\Delta}^1_\beta \cup \{k\delta + \beta\}$
e	$\{\gamma \in \widehat{\Delta}^0_{\alpha_\Sigma} \cup \widehat{\Delta}^1_{\alpha_\Sigma} : \lvert\gamma\rvert = \lvert\theta_\Sigma\rvert\} \cup \{\delta + \alpha_\Sigma\} \cup \{\delta + \alpha_\Sigma + \beta\}$
f	$\langle A(\Sigma)\rangle_\ell$
g	$\{\gamma \in \widehat{\Delta}^1_\beta : \gamma = k\delta - \tau, \tau \in \langle \Sigma_\beta\rangle\} \cup \{k\delta + \beta\}$

Theorem (Stara)

Assume $\mu \in \mathcal{M}_\sigma$ and $\alpha \in \widehat{\Delta}^+_{re}$. Then $\mathcal{I}_{\alpha,\mu} \neq \emptyset$ if and only if $\alpha \in \widehat{\Delta}_\mu$. Morever, there is a precise description of the poset structure of $\mathcal{I}_{\alpha,\mu}$.

Example: $E_6^{(1)}$, $\Pi_1 = \{\alpha_2\}$

1. Choose $\Sigma \simeq A_1 = \{\alpha_7\}$. Then $\mu = \delta - \theta_\Sigma = \theta$. Then $A(\Sigma) \cong A_5$ and
$$\mathcal{I}_{\beta,\mu} \neq \emptyset \iff \beta \in \langle A(\Sigma)\rangle \cup (\delta - \langle A(\Sigma)\rangle)$$
The $\mathcal{I}_{\beta,\mu}$ are singletons consisting just of the element of minimal length mapping β to μ.

Example: $E_6^{(1)}$, $\Pi_1 = \{\alpha_2\}$

1. Choose $\Sigma \simeq A_1 = \{\alpha_7\}$. Then $\mu = \delta - \theta_\Sigma = \theta$. Then $A(\Sigma) \cong A_5$ and
$$\mathcal{I}_{\beta,\mu} \neq \emptyset \iff \beta \in \langle A(\Sigma)\rangle \cup (\delta - \langle A(\Sigma)\rangle)$$
The $\mathcal{I}_{\beta,\mu}$ are singletons consisting just of the element of minimal length mapping β to μ.
2. Choose $\Sigma \simeq A_5 = \{\alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}$. Then $\mu = \delta - \theta_\Sigma = 2\alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_7$ is the highest root of $A(\Sigma)$, a root system of type D_5, and
$$\mathcal{I}_{\beta,\mu} \neq \emptyset \iff \beta \in \Delta^+(D_5)$$

Example: $E_6^{(1)}$, $\Pi_1 = \{\alpha_2\}$

In this case the $\mathcal{I}_{\beta,\mu}$ may have different "shapes"

$$\begin{aligned}
\mathcal{I}_{\alpha_7,\mu} = \{&\{2,4,3,5,4,2\},\{2,4,3,1,5,4,2\},\{2,4,3,5,4,2,6\},\{2,4,3,1,5,4,2,3\},\{2,4,3,1,5,4,2,6\},\\
&\{2,4,3,5,4,2,6,5\},\{2,4,3,1,5,4,2,3,4\},\{2,4,3,1,5,4,2,3,6\},\{2,4,3,1,5,4,2,6,5\},\\
&\{2,4,3,5,4,2,6,5,4\},\{2,4,3,1,5,4,2,3,4,5\},\{2,4,3,1,5,4,2,3,4,6\},\{2,4,3,1,5,4,2,3,6,5\},\\
&\{2,4,3,1,5,4,2,6,5,4\},\{2,4,3,5,4,2,6,5,4,3\}\}\\
\mathcal{I}_{\alpha_2+\alpha_7,\mu} = \{&\{2,4,3,5,4\},\{2,4,3,1,5,4\},\{2,4,3,5,4,6\},\{2,4,3,1,5,4,3\},\{2,4,3,1,5,4,6\},\{2,4,3,5,4,6,5\},\\
&\{2,4,3,1,5,4,3,6\},\{2,4,3,1,5,4,6,5\},\{2,4,3,1,5,4,3,6,5\}\}\\
\mathcal{I}_{\alpha_2+\alpha_4,\mu} = \{&\{2,4,3,5,7\},\{2,4,3,1,5,7\},\{2,4,3,5,6,7\},\{2,4,3,1,5,6,7\},\{2,4,3,1,5,4,3,6,5,4,7\}\}
\end{aligned}$$

A possibly related problem: the CDSW conjecture

Let $\mathfrak{g}$ be simple. Consider the $\mathfrak{g}$ − *map*

$$c : \mathfrak{g} \to \bigwedge^2 \mathfrak{g},\ c(x) = \sum_{i=1}^{\dim \mathfrak{g}} [x, x_i] \wedge x^i,$$

where $\{x_i\}$, $\{x^i\}$ are a pair of dual bases w.r.t. the Killing form of $\mathfrak{g}$. Set $R = \bigwedge(\mathfrak{g} \oplus \mathfrak{g})$. Using c, can define three R-valued $\mathfrak{g}$-maps,

$$c_1 : \mathfrak{g} \to \bigwedge^2 \mathfrak{g} \otimes 1,\ c_2 : \mathfrak{g} \to 1 \otimes \bigwedge^2 \mathfrak{g},\ c_3 : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}.$$

Define

$$B = R/\langle Im(c_1), Im(c_2)\rangle, \quad A = R/\langle Im(c_1), Im(c_2), Im(c_3)\rangle.$$

Cachazo-Douglas-Seiberg-Witten conjecture

If S is the image in A of $\sum_i x_i \otimes x^i$, then $A^{\mathfrak{g}} \cong \mathbb{C}[S]/(S^{h^\vee})$, $h^\vee$ being the dual Coxeter number of $\mathfrak{g}$.

The CDSW conjecture for symmetric spaces

Kumar has version of the conjecture for infinitesimal symmetric spaces $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$. Just mimic the above construction using the $\mathfrak{g}_0 -$ *map* $\tilde{c} : \mathfrak{g}_0 \to \bigwedge^2 \mathfrak{g}_1$, $\tilde{c}(x) = \sum_{i=1}^{\dim \mathfrak{g}} [x, x_i] \wedge x^i$, where $\{x_i\}$, $\{x^i\}$ are a pair of dual bases of $\mathfrak{g}_1$.Correspondingly, set
$\tilde{A} = \bigwedge(\mathfrak{g}_1 \oplus \mathfrak{g}_1)/\langle Im(\tilde{c}_1), Im(\tilde{c}_2), Im(\tilde{c}_3)\rangle$ and $\tilde{S} = \sum_{i=1}^{\dim \mathfrak{g}_1} x_i \otimes x^i$

Proposition

If $\mathfrak{g}_1$ is irreducible as a $\mathfrak{g}_0$-module, then $\tilde{A}^{\mathfrak{g}}$ is a truncated polynomial algebra generated by $\tilde{S}$.

Speculation

$$\dim \tilde{A}^{\mathfrak{g}} = \text{nilpotency class of } \tilde{S} = \min_{\alpha \in \widehat{\Pi}, \mu} (\dim(\min \mathcal{I}_{\alpha,\mu})).$$

Speculation

$$\dim \tilde{A}^{\mathfrak{g}} = \text{nilpotency class of } \tilde{S} = \min_{\alpha \in \widehat{\Pi}, \mu} (\dim(\min \mathcal{I}_{\alpha,\mu})).$$

The above statement turns out to be true

- in the adjoint case, where it reduces to the CDSW conjecture;
- in the graded case with $\mathfrak{g}_0, \mathfrak{g}_1$ simple, where it reduces to a conjecture of Kumar ;
- for any g, σ with $\mathfrak{g}_1$ irreducible such that $\dim \mathfrak{g}_1 \leqslant 18$ (MAGMA computations, done with the help of John Cannon).