Asymptotic representation theory of $\mathrm{U}(N)$ and application to Yang–Mills theory

Thibaut Lemoine

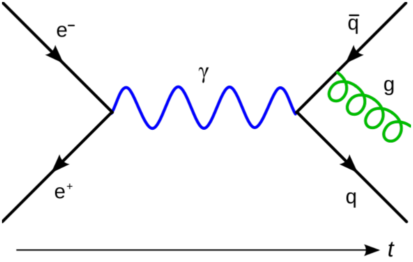
Collège de France

Closing conference of Cortipom ANR project, June 2025

Based on arXiv:2405.08393 w/ Mylène Maïda (Univ. Lille) and arXiv:2303.11286

COLLÈGE
DE FRANCE
1530

I - Two-dimensional Yang–Mills theory

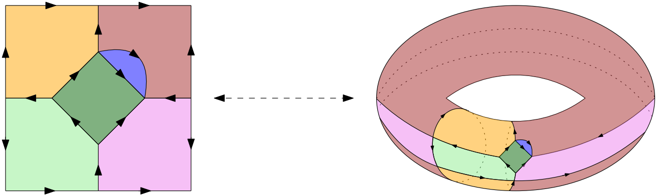


2D Yang–Mills theory

Initiated by physicists in the 70's and later ('t Hooft, Wilson, Gross, Migdal, Witten...), made rigorous by mathematicians in the 90's and later (Driver, Sengupta, Lévy).

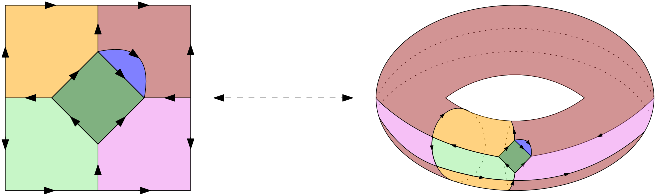
Initiated by physicists in the 70's and later ('t Hooft, Wilson, Gross, Migdal, Witten...), made rigorous by mathematicians in the 90's and later (Driver, Sengupta, Lévy).

- Spacetime : $\mathbb{G} = (\mathbf{V}, \mathbf{E}, \mathbf{F})$ oriented topological map embedded in a closed orientable surface $\Sigma = \Sigma_{g,t}$ of genus g and area t (or in $\mathbb{R}^2$).



Initiated by physicists in the 70's and later ('t Hooft, Wilson, Gross, Migdal, Witten...), made rigorous by mathematicians in the 90's and later (Driver, Sengupta, Lévy).

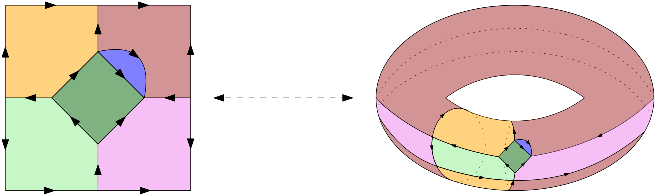
- Spacetime : $\mathbb{G} = (\mathbf{V}, \mathbf{E}, \mathbf{F})$ oriented topological map embedded in a closed orientable surface $\Sigma = \Sigma_{g,t}$ of genus g and area t (or in $\mathbb{R}^2$).



- Structure group: G compact Lie group, e.g. $\mathrm{U}(1)$, $\mathrm{SU}(2)$, $\mathrm{SU}(3)$, $\mathrm{U}(N)$...

Initiated by physicists in the 70's and later ('t Hooft, Wilson, Gross, Migdal, Witten...), made rigorous by mathematicians in the 90's and later (Driver, Sengupta, Lévy).

- Spacetime : $\mathbb{G} = (\mathbf{V}, \mathbf{E}, \mathbf{F})$ oriented topological map embedded in a closed orientable surface $\Sigma = \Sigma_{g,t}$ of genus g and area t (or in $\mathbb{R}^2$).



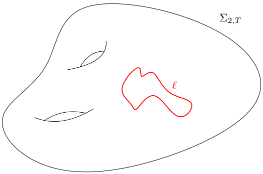
- Structure group: G compact Lie group, e.g. $\mathrm{U}(1)$, $\mathrm{SU}(2)$, $\mathrm{SU}(3)$, $\mathrm{U}(N)$...
- **Discrete** Yang–Mills measure on $G^{\mathbf{E}} = \{\omega : \mathbf{E} \to G\}$:

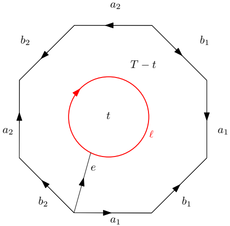
heat kernel on G at time $|f|$ | Uniform measure on $G^{\mathbf{E}}$

$$d\mu_{\mathbb{G},G,t}(\omega) = \frac{1}{Z_G(g,t)} \prod_{f \in \mathbf{F}} p_{|f|}(\Omega_\omega(f))\, d\omega$$

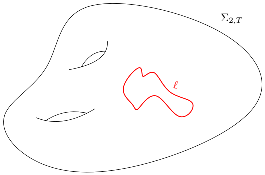
Partition function | Curvature of ω on the face f

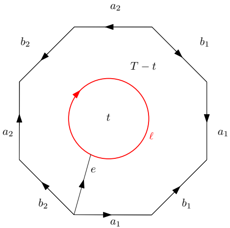
The discrete Yang–Mills measure defines a discrete G-valued Markovian random field $(H_\ell)_{\ell \in \mathrm{L}(\mathbb{G})}$. It can be extended to a **continuous** random field $(H_\ell)_{\ell \in \mathrm{L}(\Sigma)}$, the **Yang–Mills holonomy field** (Lévy 2003).





The discrete Yang–Mills measure defines a discrete G-valued Markovian random field $(H_\ell)_{\ell \in \mathrm{L}(\mathbb{G})}$. It can be extended to a **continuous** random field $(H_\ell)_{\ell \in \mathrm{L}(\Sigma)}$, the **Yang–Mills holonomy field** (Lévy 2003).

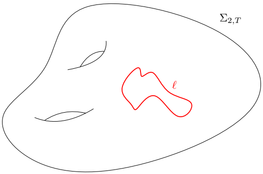


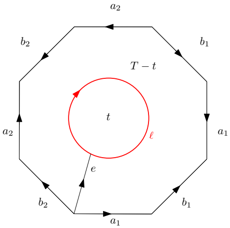


- Partition function

$$Z_G(g,t) = \int_{G^{2g}} p_t([x_1,y_1]\dots[x_g,y_g])dx_1dy_1\dots dx_gdy_g,$$

The discrete Yang–Mills measure defines a discrete G-valued Markovian random field $(H_\ell)_{\ell \in \mathrm{L}(\mathbb{G})}$. It can be extended to a **continuous** random field $(H_\ell)_{\ell \in \mathrm{L}(\Sigma)}$, the **Yang–Mills holonomy field** (Lévy 2003).





- Partition function

$$Z_G(g,t) = \int_{G^{2g}} p_t([x_1,y_1]\dots[x_g,y_g])dx_1dy_1\dots dx_gdy_g,$$

- Wilson loop expectation, for $\ell = e_1 \dots e_k$, $e_i \in \mathbf{E}$:

$$\mathbb{E}[\mathrm{tr}(H_\ell)] = \int_{G^{\mathbf{E}}} \mathrm{tr}(\omega(e_1)\dots\omega(e_k))d\mu_{\mathbb{G},G,t}(\omega).$$

Principle: take $G = \mathrm{U}(N)$ (or $\mathrm{SU}(N)$, or any compact classical group $G \subset \mathrm{GL}_N(\mathbb{C})$) and let $N \to \infty$.

Large N regime

Principle: take $G = \mathrm{U}(N)$ (or $\mathrm{SU}(N)$, or any compact classical group $G \subset \mathrm{GL}_N(\mathbb{C})$) and let $N \to \infty$.

$\longrightarrow$ Relation to random matrix theory and free probability.

Principle: take $G = \mathrm{U}(N)$ (or $\mathrm{SU}(N)$, or any compact classical group $G \subset \mathrm{GL}_N(\mathbb{C})$) and let $N \to \infty$.

$\longrightarrow$ Relation to random matrix theory and free probability.

$\longrightarrow$ **Gauge/string duality** (Gross–Taylor 1993, unsolved): partition function and Wilson loop expectations for $\mathrm{SU}(N)$ should have topological expansions in $\frac{1}{N}$ and be related to string theory.

Principle: take $G = \mathrm{U}(N)$ (or $\mathrm{SU}(N)$, or any compact classical group $G \subset \mathrm{GL}_N(\mathbb{C})$) and let $N \to \infty$.

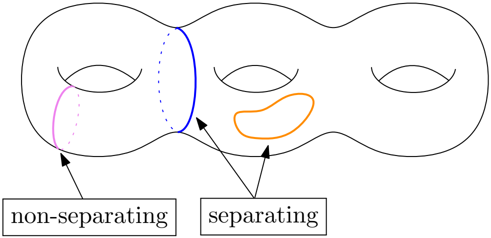
$\longrightarrow$ Relation to random matrix theory and free probability.

$\longrightarrow$ **Gauge/string duality** (Gross–Taylor 1993, unsolved): partition function and Wilson loop expectations for $\mathrm{SU}(N)$ should have topological expansions in $\frac{1}{N}$ and be related to string theory.

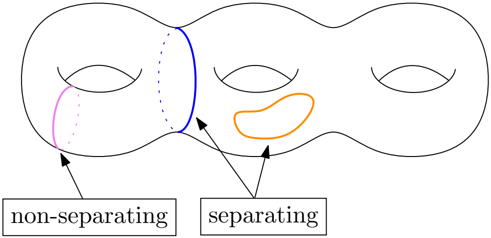
$\longrightarrow$ **Master field** (Singer 1995, almost fully solved): There is a "master field" $\Phi : \mathrm{L}(\Sigma) \to \mathbb{C}$ such that for any $\ell \in \mathrm{L}(\Sigma)$, for structure group $\mathrm{U}(N)$,

$$\lim_{N\to\infty} \mathbb{E}[\mathrm{tr}(H_\ell)] = \Phi(\ell).$$

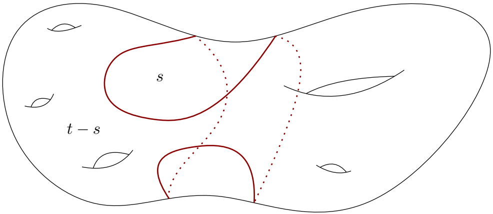
Simple loops: loops with no self-intersections. They can be separating or non-separating, depending on the surface obtained by removing the loop:



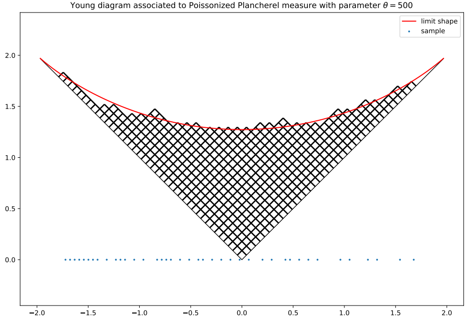
Simple loops: loops with no self-intersections. They can be separating or non-separating, depending on the surface obtained by removing the loop:



We shall focus on contractible simple loops, which are always separating. For other simple loops, it is another story (Dahlqvist–TL 2023, 2025).



II - (Asymptotic) representation theory



A **representation** of a compact Lie group G is a couple (ρ, V) where $\rho : G \to \mathrm{GL}(V)$ (continuous) group morphism.

- **character**: $\chi_\rho : g \in G \mapsto \mathrm{Tr}(\rho(g))$.
- **dimension**: $d_\rho = \dim V = \chi_\rho(1_G)$.
- (ρ, V) is **irreducible** if V is the only nontrivial invariant subspace of V for the action of ρ. The **dual** $\widehat{G}$ is the countable set of equivalence classes λ of irreducible representations of G.
- The characters χ_λ of irreducible representations are eigenfunctions of the Laplacian:
$$\Delta\chi_\lambda = -c_2(\lambda)\chi_\lambda,$$
and $c_2(\lambda) \geqslant 0$ is called the **Casimir**.

Peter–Weyl's theorem

The irreducible characters $\{\chi_\lambda, \lambda \in \widehat{G}\}$ form a Hilbert basis of the Hilbert space

$$\mathcal{H} = \{f \in L^2(G) : f(hgh^{-1}) = f(g),\ \forall g, h \in G\} \subset L^2(G).$$

Peter–Weyl's theorem

The irreducible characters $\{\chi_\lambda, \lambda \in \widehat{G}\}$ form a Hilbert basis of the Hilbert space

$$\mathcal{H} = \{f \in L^2(G) : f(hgh^{-1}) = f(g),\ \forall g, h \in G\} \subset L^2(G).$$

$\longrightarrow$ Application to the heat kernel:

$$p_t(g) = \sum_{\lambda \in \widehat{G}} e^{-\frac{t}{2} c_2(\lambda)} d_\lambda \chi_\lambda(g).$$

Peter–Weyl's theorem

The irreducible characters $\{\chi_\lambda, \lambda \in \widehat{G}\}$ form a Hilbert basis of the Hilbert space

$$\mathcal{H} = \{f \in L^2(G) : f(hgh^{-1}) = f(g),\ \forall g, h \in G\} \subset L^2(G).$$

$\longrightarrow$ Application to the heat kernel:

$$p_t(g) = \sum_{\lambda \in \widehat{G}} e^{-\frac{t}{2} c_2(\lambda)} d_\lambda \chi_\lambda(g).$$

Proposition (Migdal 1975)

For any $g \geqslant 0$ and $t > 0$,

$$Z_G(g, t) = \sum_{\lambda \in \widehat{G}} d_\lambda^{2-2g} e^{-\frac{t}{2} c_2(\lambda)}.$$

Peter–Weyl's theorem

The irreducible characters $\{\chi_\lambda, \lambda \in \widehat{G}\}$ form a Hilbert basis of the Hilbert space

$$\mathcal{H} = \{f \in L^2(G) : f(hgh^{-1}) = f(g),\ \forall g, h \in G\} \subset L^2(G).$$

$\longrightarrow$ Application to the heat kernel:

$$p_t(g) = \sum_{\lambda \in \widehat{G}} e^{-\frac{t}{2}c_2(\lambda)} d_\lambda \chi_\lambda(g).$$

Proposition (Migdal 1975)

For any $g \geqslant 0$ and $t > 0$,

$$Z_G(g,t) = \sum_{\lambda \in \widehat{G}} d_\lambda^{2-2g} e^{-\frac{t}{2}c_2(\lambda)}.$$

$\longrightarrow$ For $g = 1$: $Z_G(1,t) = \sum e^{-\frac{t}{2}c_2(\lambda)}$.

Peter–Weyl's theorem

The irreducible characters $\{\chi_\lambda, \lambda \in \widehat{G}\}$ form a Hilbert basis of the Hilbert space

$$\mathcal{H} = \{f \in L^2(G) : f(hgh^{-1}) = f(g),\ \forall g, h \in G\} \subset L^2(G).$$

$\longrightarrow$ Application to the heat kernel:

$$p_t(g) = \sum_{\lambda \in \widehat{G}} e^{-\frac{t}{2} c_2(\lambda)} d_\lambda \chi_\lambda(g).$$

Proposition (Migdal 1975)

For any $g \geqslant 0$ and $t > 0$,

$$Z_G(g, t) = \sum_{\lambda \in \widehat{G}} d_\lambda^{2-2g} e^{-\frac{t}{2} c_2(\lambda)}.$$

$\longrightarrow$ For $g = 1$: $Z_G(1, t) = \sum e^{-\frac{t}{2} c_2(\lambda)}$.

$\longrightarrow$ For $g \geqslant 2$ and $t \to 0$, $Z_G(g, 0) = \sum d_\lambda^{2-2g}$.

Unitary group: $\mathrm{U}(N) = \{U \in \mathcal{M}_N(\mathbb{C}) : UU^* = U^*U = I_N\} \subset \mathrm{GL}_N(\mathbb{C})$.

Unitary group: $\mathrm{U}(N) = \{U \in \mathcal{M}_N(\mathbb{C}) : UU^* = U^*U = I_N\} \subset \mathrm{GL}_N(\mathbb{C})$.

1. Dual:

$$\widehat{\mathrm{U}}(N) = \{\lambda = (\lambda_1, \ldots, \lambda_N) \in \mathbb{Z}^N : \ \lambda_1 \geqslant \ldots \geqslant \lambda_N\}.$$

Unitary group: $\mathrm{U}(N) = \{U \in \mathcal{M}_N(\mathbb{C}) : UU^* = U^*U = I_N\} \subset \mathrm{GL}_N(\mathbb{C})$.

1. Dual:
$$\widehat{\mathrm{U}}(N) = \{\lambda = (\lambda_1, \ldots, \lambda_N) \in \mathbb{Z}^N : \; \lambda_1 \geqslant \ldots \geqslant \lambda_N\}.$$

2. Dimension:
$$d_\lambda = \prod_{1 \leqslant i < j \leqslant N} \frac{\lambda_i - \lambda_j + j - i}{j - i}.$$

Unitary group: $\mathrm{U}(N) = \{U \in \mathcal{M}_N(\mathbb{C}) : UU^* = U^*U = I_N\} \subset \mathrm{GL}_N(\mathbb{C})$.

1. Dual:
$$\widehat{\mathrm{U}}(N) = \{\lambda = (\lambda_1, \ldots, \lambda_N) \in \mathbb{Z}^N : \ \lambda_1 \geqslant \ldots \geqslant \lambda_N\}.$$

2. Dimension:
$$d_\lambda = \prod_{1 \leqslant i < j \leqslant N} \frac{\lambda_i - \lambda_j + j - i}{j - i}.$$

3. Character:
$$\chi_\lambda(U) = \underbrace{s_\lambda(x_1, \ldots, x_N)}_{\text{Schur function}} = \frac{\det\left(x_i^{\lambda_j + N - j}\right)}{\det(x_i^{N-j})},$$

where $x_1, \ldots, x_N$ eigenvalues of $U \in \mathrm{U}(N)$.

Unitary group: $\mathrm{U}(N) = \{U \in \mathcal{M}_N(\mathbb{C}) : UU^* = U^*U = I_N\} \subset \mathrm{GL}_N(\mathbb{C})$.

1. Dual:
$$\widehat{\mathrm{U}}(N) = \{\lambda = (\lambda_1, \dots, \lambda_N) \in \mathbb{Z}^N : \ \lambda_1 \geqslant \dots \geqslant \lambda_N\}.$$

2. Dimension:
$$d_\lambda = \prod_{1 \leqslant i < j \leqslant N} \frac{\lambda_i - \lambda_j + j - i}{j - i}.$$

3. Character:
$$\chi_\lambda(U) = \underbrace{s_\lambda(x_1, \dots, x_N)}_{\text{Schur function}} = \frac{\det\left(x_i^{\lambda_j + N - j}\right)}{\det(x_i^{N-j})},$$
where $x_1, \dots, x_N$ eigenvalues of $U \in \mathrm{U}(N)$.

4. Casimir:
$$c_2(\lambda) = \frac{1}{N} \sum_{i=1}^{N} \lambda_i(\lambda_i + N + 1 - 2i) = \frac{1}{N} \langle \lambda, \lambda + 2\rho \rangle = \frac{1}{N} \left(\|\lambda + \rho\|^2 - \|\rho\|^2 \right),$$
where
$$\rho = \left(\frac{N-1}{2}, \frac{N-3}{2}, \dots, -\frac{N-3}{2}, -\frac{N-1}{2} \right).$$

For $t > 0$, set $q_t = e^{-t/2} \in (0,1)$. Distribution $\mathscr{G}_N(q_t)$ on $\widehat{\mathrm{U}}(N)$:

$$\mathbb{P}(\lambda = \mu) = \frac{1}{Z_{\mathrm{U}(N)}(1,t)} e^{-\frac{t}{2} c_2(\mu)}, \quad \forall \mu \in \widehat{\mathrm{U}}(N).$$

For $t > 0$, set $q_t = e^{-t/2} \in (0,1)$. Distribution $\mathscr{G}_N(q_t)$ on $\widehat{\mathrm{U}}(N)$:

$$\mathbb{P}(\lambda = \mu) = \frac{1}{Z_{\mathrm{U}(N)}(1,t)} e^{-\frac{t}{2} c_2(\mu)}, \quad \forall \mu \in \widehat{\mathrm{U}}(N).$$

"Discrete Gaussian" because

$$e^{-\frac{t}{2} c_2(\lambda)} = e^{-\frac{t}{2N}\left(\|\lambda+\rho\|^2 - \|\rho\|^2\right)} = \mathrm{cst} \cdot e^{-\frac{t}{2N}\|\lambda+\rho\|^2}$$

For $t > 0$, set $q_t = e^{-t/2} \in (0, 1)$. Distribution $\mathscr{G}_N(q_t)$ on $\widehat{\mathrm{U}}(N)$:

$$\mathbb{P}(\lambda = \mu) = \frac{1}{Z_{\mathrm{U}(N)}(1, t)} e^{-\frac{t}{2} c_2(\mu)}, \quad \forall \mu \in \widehat{\mathrm{U}}(N).$$

"Discrete Gaussian" because

$$e^{-\frac{t}{2} c_2(\lambda)} = e^{-\frac{t}{2N} \left(\|\lambda + \rho\|^2 - \|\rho\|^2 \right)} = \mathrm{cst} \cdot e^{-\frac{t}{2N} \|\lambda + \rho\|^2}$$

$\longrightarrow$ Case $N = 1$: integer Gaussian distribution

$$\mathbb{P}(n) = \theta(q_t) e^{-\frac{t}{2} n^2}, \quad \forall n \in \mathbb{Z},$$

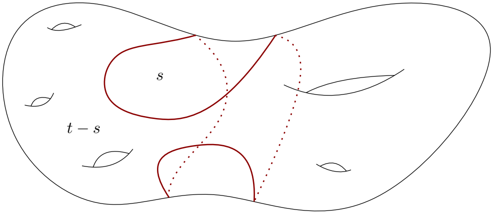
where

$$\theta(q) = \sum_{n \in \mathbb{Z}} q^{n^2}, \quad \forall q \in \mathbb{C}, |q| < 1.$$

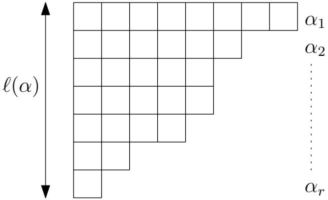
Theorem (TL 2025)

Let Σ be a compact surface of genus $g \geqslant 1$. For any contractible simple loop ℓ in $\Sigma_{g,t}$ with interior area $s \in (0,t)$, if $\lambda \sim \mathscr{G}_N(q_t)$,

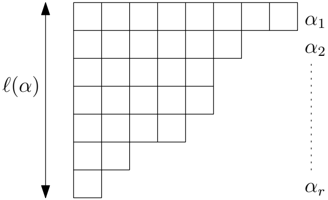
$$\mathbb{E}[\mathrm{tr}(H_\ell)] = \frac{Z_{\mathrm{U}(N)}(1,t)}{Z_{\mathrm{U}(N)}(g,t)}\mathbb{E}\left[\sum_{\mu \searrow \lambda} \frac{d_\mu}{N d_\lambda^{2g-1}} e^{-\frac{s}{2}(c_2(\mu)-c_2(\lambda))}\right].$$



An integer partition is a finite family $\alpha = (\alpha_1, \ldots, \alpha_r)$ of positive integers such that $\alpha_1 \geqslant \ldots \geqslant \alpha_r$. Set $\ell(\alpha) = r$ its **length** and $|\alpha| = \sum_i \alpha_i$ its **size**. It is represented by a **Young diagram**.



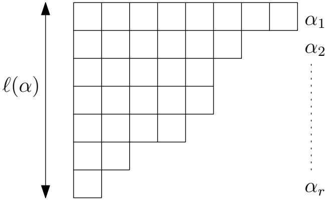
An integer partition is a finite family $\alpha = (\alpha_1, \dots, \alpha_r)$ of positive integers such that $\alpha_1 \geqslant \dots \geqslant \alpha_r$. Set $\ell(\alpha) = r$ its **length** and $|\alpha| = \sum_i \alpha_i$ its **size**. It is represented by a **Young diagram**.



- The set $\mathcal{P}_n$ of partitions of size n is in bijection with the dual of the symmetric group S_n: $\mathcal{P}_n \simeq \widehat{S}_n$.

Integer partitions

An integer partition is a finite family $\alpha = (\alpha_1, \ldots, \alpha_r)$ of positive integers such that $\alpha_1 \geqslant \ldots \geqslant \alpha_r$. Set $\ell(\alpha) = r$ its **length** and $|\alpha| = \sum_i \alpha_i$ its **size**. It is represented by a **Young diagram**.



- The set $\mathcal{P}_n$ of partitions of size n is in bijection with the dual of the symmetric group S_n: $\mathcal{P}_n \simeq \widehat{S}_n$.
- For any $n \geqslant 1$, set $p(n) = \#\mathcal{P}_n$ the number of partitions of n. The corresponding generating function satisfies

$$\sum_{n \geqslant 1} p(n) q^n = \phi(q)^{-1},$$

where

$$\phi(q) = \prod_{m=1}^{\infty} (1 - q^m).$$

Fix $q \in (0,1)$. The q-**uniform measure** $\mathscr{U}(q)$ (Vershik 1995, Bloch–Okounkov 2000) is a measure on $\mathcal{P}$ defined as follows:

1. Draw n randomly with geometric distribution of parameter $1-q$.
2. Conditionally to $|\lambda| = n$, draw λ uniformly in $\mathcal{P}_n$.

It follows that

$$\mathbb{P}(\lambda) = \phi(q) q^{|\lambda|}, \quad \forall \lambda \in \mathcal{P}.$$

Fix $q \in (0, 1)$. The q-**uniform measure** $\mathscr{U}(q)$ (Vershik 1995, Bloch–Okounkov 2000) is a measure on $\mathcal{P}$ defined as follows:

1. Draw n randomly with geometric distribution of parameter $1 - q$.
2. Conditionally to $|\lambda| = n$, draw λ uniformly in $\mathcal{P}_n$.

It follows that

$$\mathbb{P}(\lambda) = \phi(q) q^{|\lambda|}, \quad \forall \lambda \in \mathcal{P}.$$

Other well-known distribution: the Poissonized Plancherel measure, where n follows a Poisson distribution and conditionally to $|\lambda| = n$, λ follows the Plancherel distribution on $\mathcal{P}_n$.

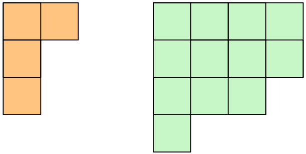
Introduce a set of couplings $\Lambda_N \subset \mathcal{P} \times \mathcal{P} \times \mathbb{Z}$, for fixed N, with a condition on lengths of partitions.

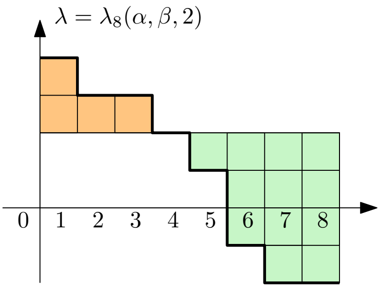
Thm (TL 2022)

There is a bijection $\lambda_N : \begin{cases} \Lambda_N & \longrightarrow & \widehat{\mathrm{U}}(N) \\ (\alpha, \beta, n) & \longmapsto & \lambda \end{cases}$ given by

$$\lambda = \lambda_N(\alpha, \beta, n) = (\alpha_1 + n, \ldots, \alpha_{\ell(\alpha)} + n, n, \ldots, n, n - \beta_{\ell(\beta)}, \ldots, n - \beta_1).$$

$\alpha = (2, 1, 1)$ $\beta = (4, 4, 3, 1)$





The bijection comes from a special version of the Schur–Weyl duality (Koike 1989) and has interesting asymptotic implications:

The bijection comes from a special version of the Schur–Weyl duality (Koike 1989) and has interesting asymptotic implications:

- (TL 2022) for any $(\alpha, \beta, n) \in \Lambda_N$,

$$c_2(\lambda_N(\alpha, \beta, n)) = |\alpha| + |\beta| + n^2 + O(N^{-1}),$$

The bijection comes from a special version of the Schur–Weyl duality (Koike 1989) and has interesting asymptotic implications:

- (TL 2022) for any $(\alpha, \beta, n) \in \Lambda_N$,

$$c_2(\lambda_N(\alpha, \beta, n)) = |\alpha| + |\beta| + n^2 + O(N^{-1}),$$

- (TL 2025) for any $(\alpha, \beta, n) \in \Lambda_N$ such that $|\alpha|, |\beta| \leqslant N^\gamma$, $\gamma \in (0, \frac{1}{3})$,

$$d_{\lambda_N(\alpha,\beta,n)} = \frac{f_\alpha f_\beta N^{|\alpha|+|\beta|}}{|\alpha|!|\beta|!} \left(1 + O(N^{3\gamma-1})\right),$$

  where f_α number of standard Young tableaux of shape α.

Let $\lambda \sim \mathscr{G}_N(q)$, and α, β, n be independent random variables such that $\alpha, \beta \sim \mathscr{U}(q)$ and $n \sim \mathscr{G}_1(q)$.

Theorem (TL–Maïda 2025)

For any measurable $f : \widehat{\mathrm{U}}(N) \to \mathbb{R}$,

$$\mathbb{E}[f(\lambda)] = \frac{\theta(q)}{\phi(q)^2}\mathbb{E}[f(\lambda_N(\alpha,\beta,n))q^{\frac{2}{N}F(\alpha,\beta,n)}\mathbf{1}_{\Lambda_N}(\alpha,\beta,n)],$$

with an explicit $F : \Lambda_N \to \mathbb{R}$.

Let $\lambda \sim \mathscr{G}_N(q)$, and α, β, n be independent random variables such that $\alpha, \beta \sim \mathscr{U}(q)$ and $n \sim \mathscr{G}_1(q)$.

Theorem (TL–Maïda 2025)

For any measurable $f : \widehat{\mathrm{U}}(N) \to \mathbb{R}$,

$$\mathbb{E}[f(\lambda)] = \frac{\theta(q)}{\phi(q)^2}\mathbb{E}[f(\lambda_N(\alpha, \beta, n))q^{\frac{2}{N}F(\alpha,\beta,n)}\mathbf{1}_{\Lambda_N}(\alpha, \beta, n)],$$

with an explicit $F : \Lambda_N \to \mathbb{R}$.

$\longrightarrow$ The large N asymptotics of $\mathscr{G}_N(q)$ is given by $\mathscr{U}(q) \otimes \mathscr{U}(q) \otimes \mathscr{G}_1(q)$ (asymptotic decoupling), and the speed of convergence is controlled by deviation inequalities of $\mathscr{U}(q)$.

III - Applications

Theorem (TL–Maïda 2025)

For any $t > 0$, there are coefficients $(a_k(t))_{k \geqslant 0}$ such that for any $p \geqslant 0$,

$$Z_{\mathrm{U}(N)}(1,t) = a_0(t) + \frac{a_1(t)}{N^2} + \ldots + \frac{a_p(t)}{N^{2p}} + O(N^{-2p-2}),$$

and the coefficients $a_k(t)$ have explicit expressions in terms of Hurwitz numbers and theta functions.

Theorem (TL–Maïda 2025)

For any $t > 0$, there are coefficients $(a_k(t))_{k\geqslant 0}$ such that for any $p \geqslant 0$,

$$Z_{\mathrm{U}(N)}(1,t) = a_0(t) + \frac{a_1(t)}{N^2} + \ldots + \frac{a_p(t)}{N^{2p}} + O(N^{-2p-2}),$$

and the coefficients $a_k(t)$ have explicit expressions in terms of Hurwitz numbers and theta functions.

$\longrightarrow$ The asymptotic expansion is also a **topological expansion:** it solves (partially) the gauge/string duality conjectured by Gross and Taylor (1993).

Theorem (TL–Maïda 2025)

For any $t > 0$, there are coefficients $(a_k(t))_{k \geqslant 0}$ such that for any $p \geqslant 0$,

$$Z_{\mathrm{U}(N)}(1, t) = a_0(t) + \frac{a_1(t)}{N^2} + \ldots + \frac{a_p(t)}{N^{2p}} + O(N^{-2p-2}),$$

and the coefficients $a_k(t)$ have explicit expressions in terms of Hurwitz numbers and theta functions.

$\longrightarrow$ The asymptotic expansion is also a **topological expansion:** it solves (partially) the gauge/string duality conjectured by Gross and Taylor (1993).

$\longrightarrow$ Upcoming work (TL–Maïda 2025+): more explicit links with random surfaces and string theory (Gromov–Witten invariants)

Application 2: speed of convergence of observables

Theorem (TL 2025)

Let Σ be a compact surface of genus $g \geqslant 1$, and fix $t > s \geqslant 0$. As $N \to \infty$, we have:

- If $g \geqslant 2$:

$$Z_{\mathrm{U}(N)}(g,t) = \theta(q_t) + O(N^{2-2g}), \quad \mathbb{E}[\mathrm{tr}(H_\ell)] = q_s + O(N^{-2}).$$

- If $g = 1$, for any $\varepsilon > 0$:

$$Z_{\mathrm{U}(N)}(1,t) = \frac{\theta(q_t)}{\phi(q_t)^2} + O(N^{-2}), \quad \mathbb{E}[\mathrm{tr}(H_\ell)] = q_s + O(N^{-1+\varepsilon}).$$