Crystal skeletons: Combinatorics and axioms

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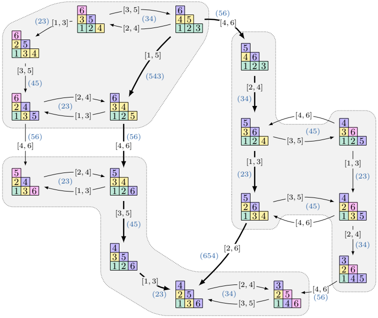

Le Croisic, France
June 9, 2025

Based on joint work with

Sarah Brauner, Sylvie Corteel, and Zajj Daugherty
preprint arXiv:2503.14782

Motivation

F_α – Gessel quasisymmetric functions indexed by compositions α

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s_λ – Schur functions indexed by partitions λ

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Goal

f symmetric function

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s_λ – Schur functions indexed by partitions λ

Goal

f symmetric function
Given quasisymmetric expansion

$$f = \sum_{\alpha \models n} c_\alpha F_\alpha$$

find the Schur expansion

$$f = \sum_{\lambda \vdash n} b_\lambda s_\lambda$$

Motivation

Quasisymmetric expansions are known in various settings, where Schur expansions are still elusive:

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- LLT polynomials
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- plethysm of Schur functions
- characters of higher Lie modules (Thrall's problem)

History

On symmetric functions

- 2010 Egge, Loehr, Warrington

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$$f = \sum_{\alpha \models n} c_\alpha F_\alpha \longrightarrow f = \sum_{\alpha \models n} c_\alpha s_\alpha$$

plus straightening

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  plus straightening

- 2019 Gessel gave sign-reversing proof of this formula

History

Using graphs and representation theory

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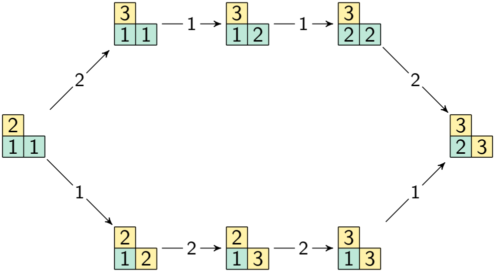
- 2023 Maas-Gariépy quasicrystals and crystal skeletons

Crystals

Combinatorial representation theory of $\mathfrak{sl}_n$ (Kashiwara – Abel prize!!!)

Crystal: $B(2,1)$



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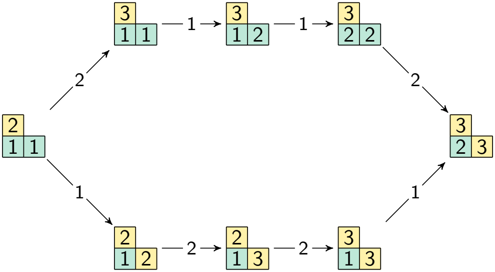

Schur function $s_{(2,1)}$

Quasi-crystals

Group tableaux by standardization – labeled by descent composition

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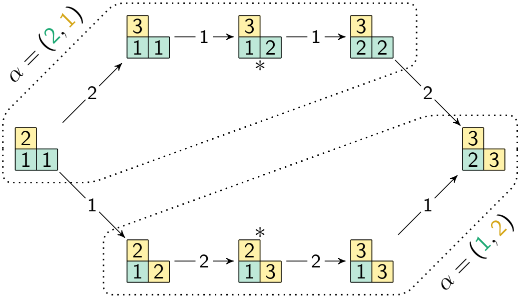

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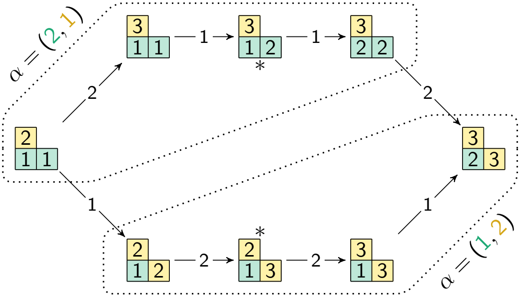
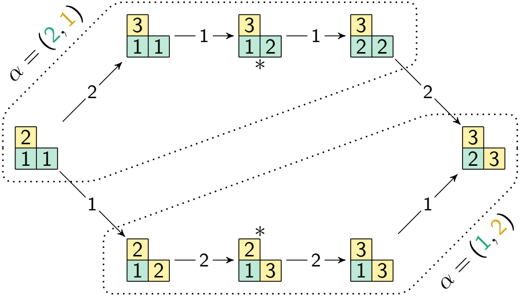


Schur function $s_{(2,1)} = F_{(2,1)} + F_{(1,2)}$

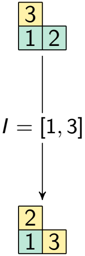
Crystal skeletons

Contract quasi-crystals to a point

Crystal: $B(2,1)$



Crystal skeleton: $\mathrm{CS}(2,1)$



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Gessel's quasisymmetric functions (1984)

$$F_\alpha = \sum_{\substack{\beta \preccurlyeq \alpha \\ \text{refinement}}} M_\beta \qquad \text{with} \qquad M_\beta = \sum_{i_1 < i_2 < \cdots < i_\ell} x_{i_1}^{\beta_1} x_{i_2}^{\beta_2} \cdots x_{i_\ell}^{\beta_\ell}$$

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Example

$$\begin{aligned} F_{(1,2,1)} =& M_{(1,2,1)} + M_{(1,1,1,1)} \\ =& x_1 x_2^2 x_3 + x_1 x_2^2 x_4 + x_2 x_3^2 x_4 + \cdots \\ &+ x_1 x_2 x_3 x_4 + x_1 x_2 x_3 x_5 + \cdots \end{aligned}$$

Schur functions

SYT(λ) set of standard Young tableaux of shape λ
$T \in \mathrm{SYT}(\lambda)$ (French notation), $\lambda \vdash n$

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Definition

i is a descent in T if $i+1$ is in higher row of the tableau
$d_1 < d_2 < \cdots < d_k$ descents in T

$$\mathrm{Des}(T) = (d_1, d_2 - d_1, \ldots, d_k - d_{k-1}, n - d_k)$$

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Example

$$T = \begin{array}{|c|c|c|c|}\hline 7 & & & \\ \hline 3 & 5 & & \\ \hline 1 & 2 & 4 & 6 \\ \hline \end{array} \qquad \text{descents}\{2,4,6\} \quad \mathrm{Des}(T) = (2,2,2,1)$$

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7			
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Theorem (Gessel 1984)

$$s_\lambda = \sum_{T \in \mathrm{SYT}(\lambda)} F_{\mathrm{Des}(T)}$$

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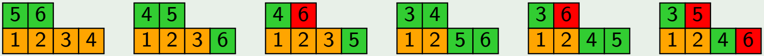
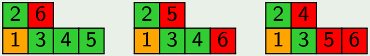
Example

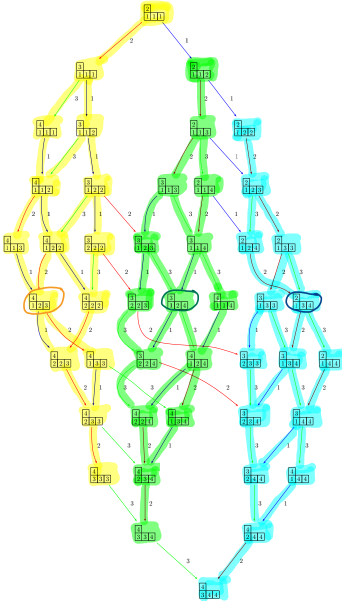
$$s_{(4,2)} = F_{(4,2)} + F_{(3,3)} + F_{(3,2,1)} + F_{(2,4)} + F_{(2,3,1)} + F_{(2,2,2)}$$



$$+ F_{(1,4,1)} + F_{(1,3,2)} + F_{(1,2,3)}$$

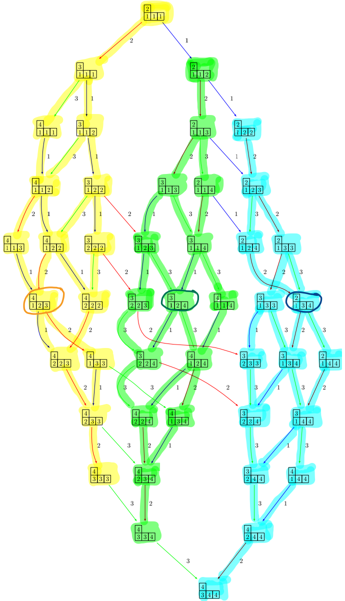


Quasicrystal



Quasicrystal components: All tableaux which standardize to the same $T \in \mathrm{SYT}(\lambda)$

Quasicrystal



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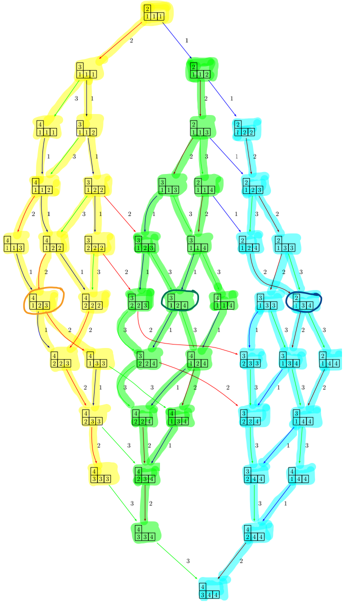

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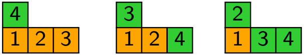
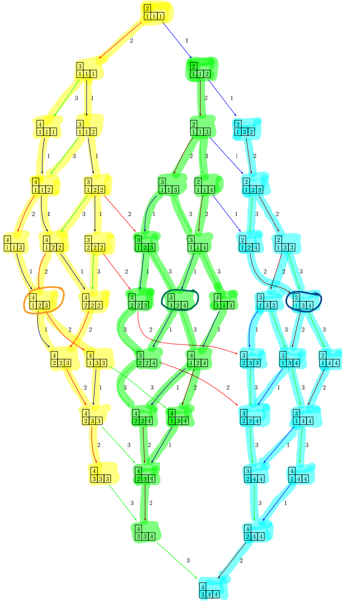


quasi-Yamanouchi tableaux



quasi-Yamanouchi tableau:

highest element in quasicrystal component

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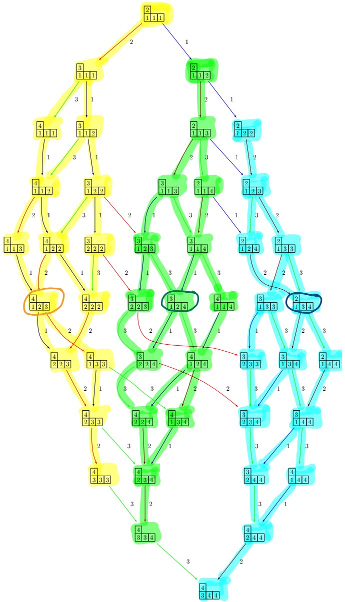

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Characterization:
Assaf, Searles 2018, Wang 2019
when i appears, some instance of i is higher than some instance of $i-1$ for all i

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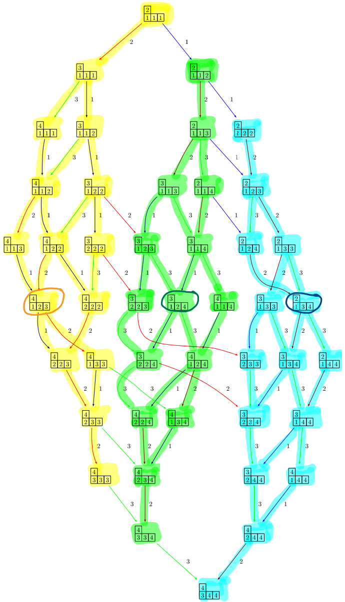

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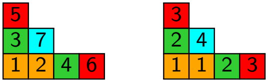
QYT(λ) is set of quasi-Yamanouchi tableaux of shape λ

quasi-Yamanouchi tableaux II

Standard tableaux and quasi-Yamanouchi tableaux correspond via (de)standardization:



Bijection

$$\text{destandard}\colon \mathsf{SYT}(\lambda) \to \mathsf{QYT}(\lambda)$$

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Results: Combinatorics of crystal skeleton

Crystal skeleton $\mathrm{CS}(\lambda)$

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- **Vertices**
  1. Standard tableaux T
  2. Descent compositions α

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- **Vertices**
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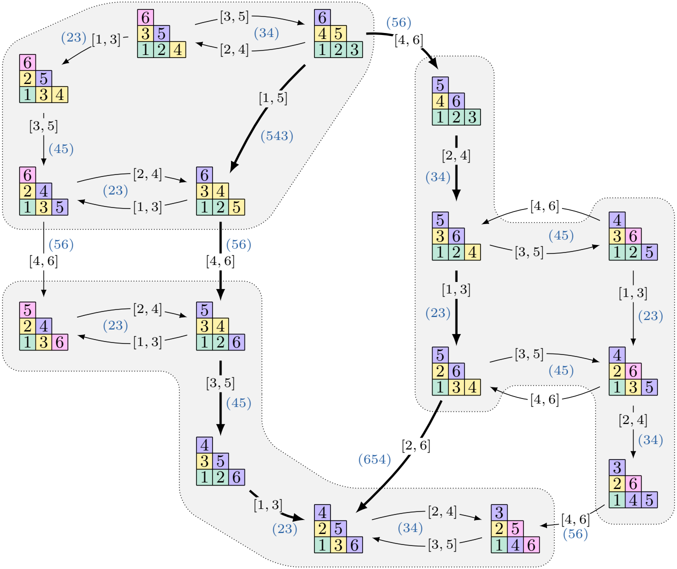
- **Edges**
  1. Dyck pattern intervals: odd length intervals I with $|I| \geqslant 3$
  2. Cycles

Results: Combinatorics of crystal skeleton



Edges: Dyck pattern intervals

Definition

- $\pi \in S_n$ permutation
- $I = [i, i+2m] \subseteq [n]$ interval of length $2m+1 \geqslant 3$

The interval I is a Dyck pattern interval of π if

$$P(\pi|_I) =$$

$i+m+1$	$i+m+2$	$\cdots$	$i+2m$	
i	$i+1$	$\cdots$	$i+m-1$	$i+m$

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Example

$\pi = 524136$. The interval $I = [2,4]$ is a Dyck pattern interval since $\pi|_{[2,4]} = 243$ and

$$P(\pi|_{[2,4]}) = \begin{array}{|c|c} \hline 4 & \multicolumn{1}{|c}{} \\ \hline 2 & \multicolumn{1}{c|}{3} \\ \hline \end{array}$$

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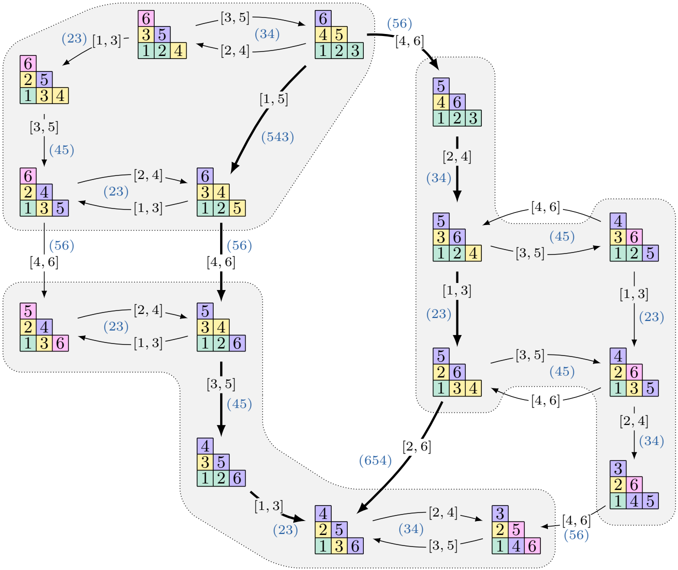

Edges: Cycles

Definition

$$\text{cycle}(\pi|_I) := (m + \pi_p, m + \pi_p - 1, \ldots, \pi_p)$$

π_p is letter where f_i acts in destandardization and $I = [i, i + 2m]$

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$$T = \begin{array}{|c|c|c|}\hline 6 & & \\ \hline 4 & 5 & \\ \hline 1 & 2 & 3 \\ \hline\end{array} \qquad \mathrm{destand}(T) = \begin{array}{|c|c|c|}\hline 6 & & \\ \hline 2 & 2 & \\ \hline 1 & 1 & 1 \\ \hline\end{array} \qquad \pi = 645123$$

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Crystal operator f_1 acts on rightmost 1 and hence $\pi_p = 3$

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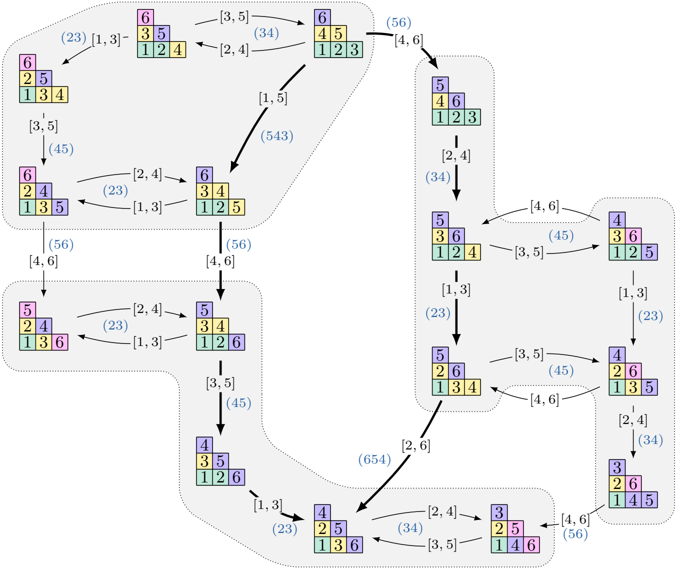
Dyck pattern interval $I = [1, 5]$

Cycle $\text{cycle}(\pi|_I) = (5, 4, 3)$

$$(5, 4, 3) \cdot T = T'$$

Edges: Cycles



Length of descent composition

Theorem (BCDS'25)

Edge in $\mathrm{CS}(\lambda)$

$$(T,\alpha) \xrightarrow{\;I\;} (T',\beta)$$

α *of length* $\ell \Rightarrow \beta$ *of length* $\ell-1, \ell$ *or* $\ell+1$

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$$I = [i, i+2m] = I^- \cup \{i+m\} \cup I^+, \qquad I^- \cup \{i+m\} \subseteq \alpha^{(j)}$$

$$\beta = (\alpha^{(1)}, \ldots, \alpha^{(j-2)}, \underbrace{\qquad\circledast\qquad} \alpha^{(j+2)}, \ldots, \alpha^{(\ell)})$$

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$$\ell: \quad \circledast = \left(\alpha^{(j-1)},\ \alpha^{(j)} \setminus \{i+m\},\ \alpha^{(j+1)} \cup \{i+m\}\right)$$

$$\ell+1: \quad \circledast = \left(\alpha^{(j-1)},\ \alpha^{(j)} \setminus \{i+m\},\ I^+ \cup \{i+m\},\ \alpha^{(j+1)} \setminus I^+\right)$$

$$\ell-1: \quad \circledast = \left(\alpha^{(j-1)} \cup I^-,\ \alpha^{(j+1)} \cup \{i+m\}\right)$$

Properties: S_n-branching

Theorem (BCDS'25)

$$\mathrm{CS}(\lambda)_{[1,n-1]} \cong \bigoplus_{\lambda^-} \mathrm{CS}(\lambda^-)$$

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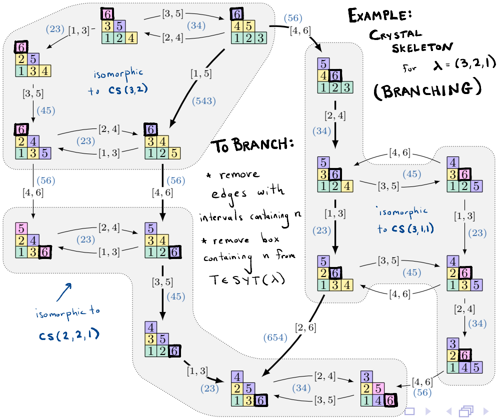

Properties: Subcrystal

Theorem (BCDS'25)

CS(λ) *contains crystal* $B(\lambda)$ *as a subgraph.*

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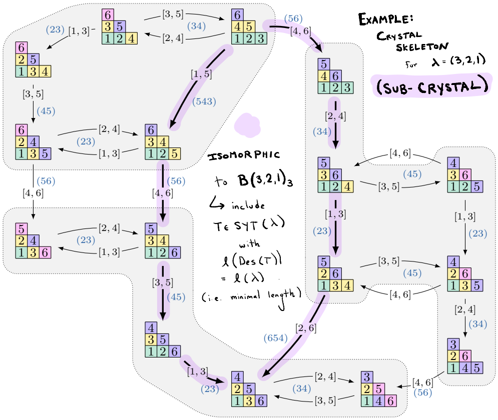

Properties: Dual equivalence subgraph

Theorem (BCDS'25 proving conjecture of Maas-Gariépy)

The *dual equivalence graph* $\mathrm{DE}(\lambda)$ *is a subgraph of* $\mathrm{CS}(\lambda)$.

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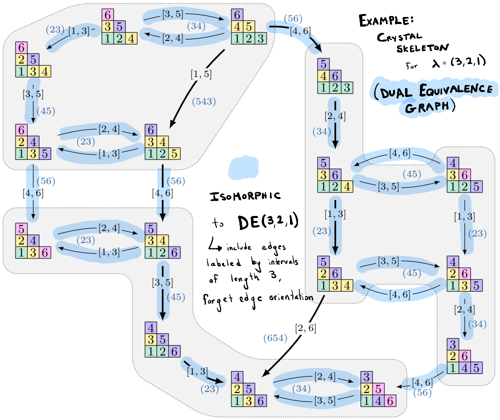

Properties: Lusztig involution

Theorem (BCDS'25)

$\mathsf{CS}(\lambda)$ *is symmetric under Lusztig involution:* $\mathsf{CS}(\lambda) \cong \mathcal{L}_n(\mathsf{CS}(\lambda))$.

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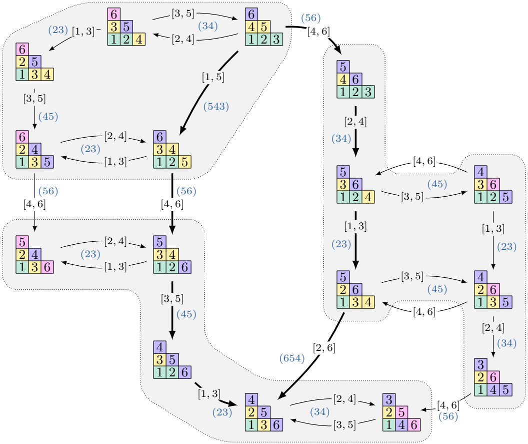
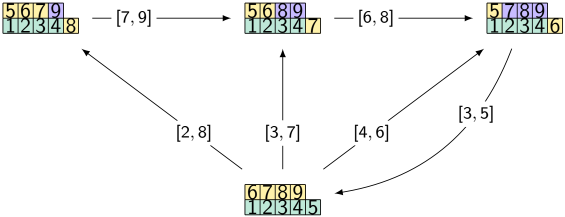


Properties: Fans

Fans for length increasing edges:



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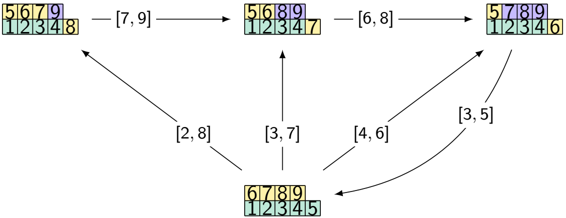

- Nested intervals up

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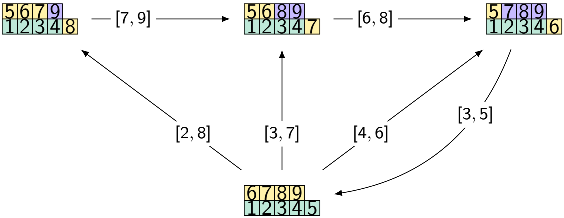
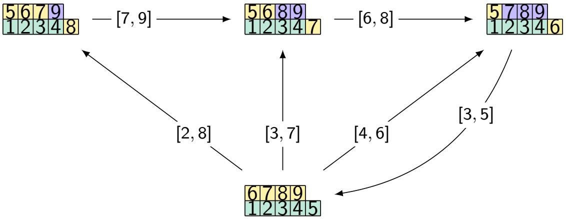


- Nested intervals up
- Length 3 intervals across

Properties: Fans

Fans for length increasing edges:



- Nested intervals up
- Length 3 intervals across
- Length 3 loop on the right

Properties: Fans

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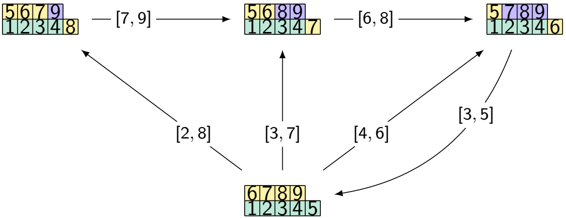

- Nested intervals up
- Length 3 intervals across
- Length 3 loop on the right

Similarly for length decreasing edges.

Axiomatic description

1. GL_n-**axioms**:
   - Strong Lusztig involution: $G \cong \mathcal{L}_n(G)$ and $G_{[1,n-1]} \cong \mathcal{L}_{n-1}(G_{[1,n-1]})$
   - Subcrystal
   - Fans

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2. S_n-**axioms**:
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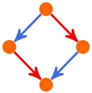
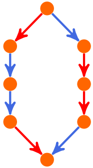
3. **Local axioms**:
   - Commutation relations (à la Stembridge): triangles, squares, pentagons, octagons + fans

Local characterization of crystals

Local characterization of simply-laced crystals associated to representations (Stembridge 2003)





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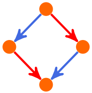
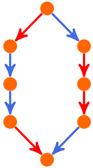
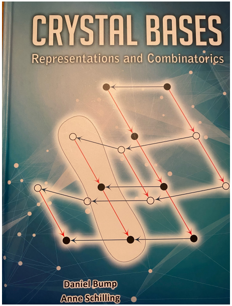
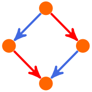


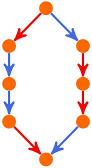


Combinatorial theory of crystals without quantum groups:

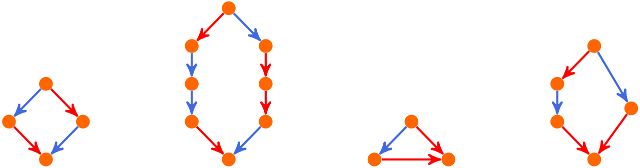


Local characterization of crystal skeletons





Local characterization of crystal skeletons



+ fans

Applications

Applications

Thank you!