A discrete-time Matsumoto–Yor theorem

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Plan of the presentation

1. Background and literature
   - Pitman's theorem
   - Matsumoto–Yor's theorem and extension
2. A discrete-time Matsumoto–Yor theorem
   - Random walk on a subgroup of SL_2
   - Main theorems obtained

Pitman's theorem (1975)

- Let $(B_t)_{t\geq 0}$ be a one-dimensional Brownian motion, then

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A Bessel process of dimension 3 = "**Brownian motion conditioned to stay forever positive**" using Doob's transform.

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Matsumoto–Yor's theorem (2000)

$$Z_t^{(\mu)} := e^{B_t^{(\mu)}} \int_0^t e^{-2B_s^{(\mu)}} ds$$

is a diffusion process on $\mathbb{R}_+$ with infinitesimal generator given by :

$$\frac{1}{2} z^2 \frac{d^2}{dz^2} + \left[\left(\frac{1}{2} + \mu \right) z + \left(\frac{K_{1-\mu}}{K_\mu} \right) \left(\frac{1}{z} \right) \right] \frac{d}{dz}$$

where K_μ is a Macdonald function (modified Bessel function of the second kind).

H. Matsumoto and M. Yor. *An analogue of Pitman's $2M - X$ theorem for exponential Wiener functionals. I. A time-inversion approach*. Nagoya Math. J. 159 125 - 166, 2000.

Laplace's principle

Let $A \subset \mathbb{R}^d$ and $\varphi : \mathbb{R}^d \to \mathbb{R}$ be a continuous function such that :

$$\int_A e^{-\varphi(x)} dx < +\infty.$$

Then :

$$\lim_{\theta \to +\infty} \frac{1}{\theta} \log \left(\int_A e^{-\theta \varphi(x)} dx \right) = -\inf_{x \in A} \varphi(x).$$

Extension of the Matsumoto–Yor theorem

The Matsumoto–Yor theorem can be interpreted through $(\mathbf{B}_t)_{t\geq 0}$:

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O'Connell and then Chhaibi extended the Matsumoto–Yor theorem in the context of semisimple Lie groups.

Random walk on a subgroup of SL_2

C. Hérent. *A discrete-time Matsumoto–Yor theorem*. Preprint 2024. (hal-04683351), arXiv :2409.01044. Accepted for publication in ESAIM : Probability and Statistics.

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Let $\gamma := (\gamma_n)_{n\in\mathbb{N}}$ be a family of i.i.d. random variables and let $\delta \in \mathbb{R}^*$ be a deterministic parameter. Let $(X_n)_{n\in\mathbb{N}}$, $(Z_n)_{n\in\mathbb{N}}$ be two processes such as :

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$$\begin{pmatrix} X_{n+1} & 0 \\ Z_{n+1} & X_{n+1}^{-1} \end{pmatrix} = \begin{pmatrix} X_n & 0 \\ Z_n & X_n^{-1} \end{pmatrix} \begin{pmatrix} \gamma_n & 0 \\ \delta & \gamma_n^{-1} \end{pmatrix}$$

with,

$$\begin{pmatrix} X_0 & 0 \\ Z_0 & X_0^{-1} \end{pmatrix} = I_2.$$

By recurrence, we have the formula for $n \in \mathbb{N}$:

$$\begin{pmatrix} X_n & 0 \\ Z_n & X_n^{-1} \end{pmatrix} = \begin{pmatrix} \prod_{i=0}^{n-1} \gamma_i & 0 \\ \delta \sum_{k=0}^{n-1} \prod_{i=0}^{k-1} \gamma_i^{-1} \prod_{j=k+1}^{n-1} \gamma_j & \prod_{i=0}^{n-1} \gamma_i^{-1} \end{pmatrix}.$$

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In the following, we will assume (except for the continuous limit of the random walk) that $\delta = 1$ to simplify certain expressions.

Discrete analogue of the Matsumoto–Yor theorem

For the common law of γ_i, we take a $GIG(\lambda, a, a)$ distribution.

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For the common law of γ_i, we take a $GIG(\lambda, a, a)$ distribution. The probability density function for this distribution is given by :

$$f(x) := \frac{1}{2K_\lambda(a^2)} x^{\lambda-1} e^{-\frac{a^2}{2}(x+\frac{1}{x})} \mathbb{1}_{x>0}$$

where $a > 0$, $\lambda \in \mathbb{R}$ and K_λ is the Macdonald function.

Discrete analogue of the Matsumoto–Yor theorem

Theorem (Markov property of the process Z)

• The process $(Z_n)_{n\in\mathbb{N}^*}$ is a homogeneous Markov chain from $Z_1 = 1$ with transition kernel given, for $x > 0$, by :

$$Q(x, dy) = \left(\frac{1}{2K_\lambda(a^2)}\right) \frac{K_\lambda\left(\frac{a^2}{y}\right)}{K_\lambda\left(\frac{a^2}{x}\right)} \frac{1}{y} e^{-\frac{a^2(x^2+y^2+1)}{2xy}} \mathbb{1}_{\mathbb{R}_+^*}(y)dy.$$

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• In addition, for all $n \in \mathbb{N}^*$:

$$\mathbb{P}\left(X_n \in dx | Z_n, \cdots, Z_1\right) = \Lambda(Z_n, dx) \text{ a.s.}$$

with, for $z > 0$: $\Lambda(z, dx) = \frac{1}{2K_\lambda\left(\frac{a^2}{z}\right)} x^{\lambda-1} e^{-\frac{a^2}{2z}\left(x+\frac{1}{x}\right)} \mathbb{1}_{\mathbb{R}_+^*}(x)dx.$

Intertwining relation and Markov property

Theorem

Let $(\gamma_n)_{n\in\mathbb{N}}$ be a sequence of i.i.d. random variables with a density function $\mathcal{C}^1$ supported on $\mathbb{R}_+$. Then, there exists a family of probability measures $(\Lambda(z,\cdot))_{z>0}$ such that, for all $n \in \mathbb{N}^*$:

$$\mathbb{P}\left(X_n \in dx | Z_n, \cdots, Z_1\right) = \Lambda(Z_n, dx) \text{ p.s.}$$

if and only if the sequence $(\gamma_n)_{n\in\mathbb{N}}$ is distributed with a $GIG(\lambda, a, a)$ law with parameters $a > 0$ and $\lambda \in \mathbb{R}$.

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$$\int_0^{+\infty} e^{-2B_s^{(\mu)}} ds \overset{\text{law}}{=} \frac{1}{2\xi}$$

where ξ is a random variable with distribution $Gamma(\mu)$, that is, with probability density $\frac{1}{\Gamma(\mu)} x^{\mu-1} e^{-x} \mathbb{1}_{x>0}$.

Discrete-time analogue of the Dufresne identity

Theorem

For $\lambda > 0$, the distribution of the random variable

$$N_\infty^{(\lambda)} = \sum_{k=0}^{+\infty} \gamma_k^{-1} \left(\prod_{i=0}^{k-1} \gamma_i^{-1} \right)^2$$

is given by the inverse-gamma density distribution π defined by :

$$\pi(x) = \frac{a^{2\lambda}}{2^\lambda \Gamma(\lambda)} x^{-\lambda-1} e^{-\frac{a^2}{2x}} \mathbb{1}_{x>0}$$

where Γ is Euler's gamma function.

Convergence of discrete processes to their continuous analogues

We have the following convergence for the random walk on the group with $\delta = \delta_n$ such that $\delta_n := \frac{1}{n}$ et $\gamma_i^{(\sqrt{n})} \sim GIG(\lambda, \sqrt{n}, \sqrt{n})$ when $n \to +\infty$:

$$\begin{pmatrix} X_{\lfloor nt \rfloor} & 0 \\ Z_{\lfloor nt \rfloor} & X_{\lfloor nt \rfloor}^{-1} \end{pmatrix} \overset{\text{law}}{\to} \begin{pmatrix} e^{B_t^{(\lambda)}} & 0 \\ e^{B_t^{(\lambda)}} \int_0^t e^{-2B_s^{(\lambda)}} ds & e^{-B_t^{(\lambda)}} \end{pmatrix}.$$

Thank you for your attention !

Appendix : New characterisation of GIG laws

$$X_2 = \gamma_0\gamma_1,\ X_3 = \gamma_0\gamma_1\gamma_2,\ Z_2 = \gamma_0^{-1} + \gamma_1,\ Z_3 = \gamma_0^{-1}\gamma_1^{-1} + \gamma_0^{-1}\gamma_2 + \gamma_1\gamma_2.$$

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Theorem (Characterisation of GIG laws)

Let $\gamma_0, \gamma_1, \gamma_2$ be three i.i.d. random variables with a density function $\mathcal{C}^1$ supported on $\mathbb{R}_+$. If the following conditional laws are equal, for $z, u > 0$:

$$\mathcal{L}\left(X_3 | Z_3 = z, Z_2 = u\right) = \mathcal{L}\left(X_2 | Z_2 = z\right),$$

then, γ_0 is distributed according to a law $GIG(\lambda, a, a)$ with $a > 0$ and $\lambda \in \mathbb{R}$.

Appendix : Intertwining relation and Markov property

Let P and Q be two Markov kernels on the measurable spaces $(E, \mathcal{E})$ and $(F, \mathcal{F})$ respectively. Consider a Markov kernel Λ from F to E, that is, an application :

$$\Lambda : (u, A) \mapsto \Lambda(u, A) \text{ with } u \in F \text{ and } A \in \mathcal{E}$$

such that for all $u \in F$, $\Lambda(u, \cdot)$ is a probability measure on E, and for all $A \in \mathcal{E}$, $\Lambda(\cdot, A)$ belongs to the space of bounded measurable functions on F.

Appendix : Intertwining relation and Markov property

Definition (Intertwining)

We will then say that the Markov kernels P and Q are intertwined by Λ if we have the relation :

$$\Lambda P = Q\Lambda$$

where the kernel composition is defined by :

$$\Lambda P(u, dv) := \int_E P(y, dv)\Lambda(u, dy).$$

We will say that two semigroups of Markovian kernels are intertwined if this relationship occurs, with the same Λ, for all times.

Appendix : Intertwining relation and Markov property

Rogers–Pitman criterion

Let $\phi : E \to F$ be a measurable function, we define the kernel Φ such that : $\Phi f := f \circ \phi$. Let Λ be a Markov kernel from F to E verifying the two conditions :

(a) $\Lambda\Phi = I$, the identity kernel on F.

(b) For all $t \geq 0$, the Markov kernel from F to F, $Q_t := \Lambda P_t \Phi$ satisfies the identity : $\Lambda P_t = Q_t \Lambda$.

Let X be a Markov process with semigroup $(P_t)_{t\geq 0}$ and initial distribution $\lambda := \Lambda(u, \cdot)$ where $u \in F$. Then, for all $t \geq 0$ and $A \in \mathcal{E}$:

$$\mathbb{P}\left(X_t \in A | \phi \circ X_s,\ 0 \leq s \leq t\right) = \Lambda(\phi \circ X_t, A) \text{ a.s.}$$

Moreover, $\phi \circ X$ is Markov with initial state u and semigroup $(Q_t)_{t\geq 0}$.