Markov duality for interacting particle systems

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Work based on

- FC, Rouven Frassek and Cristian Giardinà, *Duality for the multispecies stirring process with open boundaries*, 2024 J. Phys. A: Math. Theor. 57 295001.
- Cristian Giardinà and Frank Redig, *Duality for Markov processes: a Lie algebraic approach*, 2025+

Interacting particle system

- Continuous time Markov process
- Configuration $\eta(t) = (\eta_1(t), \ldots, \eta_L(t))$
- Infinitesimal generator $\mathcal{L}$

Examples:

- Independent random walker
- Simple Symmetric Exclusion Process

Markov Duality

Why? "From many to few". Steady state.

Definition of stochastic duality

Let $(\eta(t))_{t\geqslant 0}$ a Markov process with state space Ω, and let $(\xi(t))_{t\geqslant 0}$ be a Markov process with state space $\tilde{\Omega}$. We say that the two process are in a duality relation with duality function $D:\Omega\times\tilde{\Omega}\to\mathbb{R}$ if

$$\mathbb{E}^{\eta}\left[D(\eta(t),\xi)\right]=\mathbb{E}^{\xi}\left[D(\eta,\xi(t))\right]\qquad\forall\eta\in\Omega,\ \forall\xi\in\tilde{\Omega}$$

$\mathcal{L}$ original generator, $\mathcal{L}^{\text{dual}}$ dual generator

$$\left(\mathcal{L}D(\cdot,\xi)\right)(\eta)=(\mathcal{L}^{\text{dual}}D(\eta,\cdot))(\xi)$$

Lie algebra generators = elementary building blocks $\mathcal{L}$ and $\mathcal{L}^{\text{dual}}$.

"Find duality between representations of Lie algebras"

Abstract duality and properties

- $\mathfrak{g}$ Lie algebra
- Pick $\rho : \mathfrak{g} \to gl(V)$. $a \in \mathfrak{g}$ in reps ρ
- Pick $\hat{\rho} : \mathfrak{g} \to gl(\hat{V})$. $\hat{a} \in \mathfrak{g}$ in reps $\hat{\rho}$.

a and $\hat{a}$ duality relation if $\exists\ D \in \mathrm{End}(V \otimes \hat{V})$ such that

$$(a \otimes I_{\hat{V}})D = (I_V \otimes \hat{a})D$$

Notation: $a \xrightarrow{D} \hat{a}$.

Properties:

- $a \xrightarrow{D} \hat{a},\ b \xrightarrow{D} \hat{b}$ then
$$[a, b] \xrightarrow{D} [\hat{b}, \hat{a}]$$
- *New dualities by symmetries:* $a \xrightarrow{D} \hat{a}$ and $[a, Q] = 0$, then $a \xrightarrow{D'} \hat{a}$ with $D' = DQ$.

Recipe to construct duality

$\mathfrak{g}$ Lie algebra

- $(J_i)_{i\in\{1,\dots,n\}}$ is reps. Let

$$\mathcal{L} = J_1 J_2 \cdots J_m$$

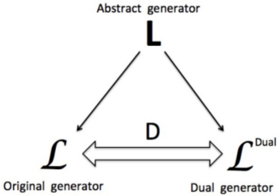
- $(K_i)_{i\in\{1,\dots,n\}}$ conjugate reps; d such that

$$J_i \xrightarrow{d} K_i \qquad \forall i$$

- Dual generator

$$\mathcal{L}^{\text{dual}} = K_m K_{m-1} \cdots K_1$$

- New duality by symmetry



Example 1: Wright Fisher-Kingman coalescent

Wright-Fisher

$$\mathcal{L} = \frac{1}{2}x(1-x)\frac{d^2}{dx^2}$$

Heisenberg Lie algebra ($\mathfrak{h}$, commutators $[\mathcal{A}, \mathcal{A}^\dagger] = I$)

$$Af(x) = f'(x), \quad A^\dagger f(x) = xf(x) \ : \ [A, A^\dagger] = I$$

$$\mathcal{L} = \frac{1}{2}A^\dagger(I - A^\dagger)A^2$$

Conjugate Lie algebra

$$af(n) = nf(n-1), \quad a^\dagger f(n) = f(n+1) \ : \ [a, a^\dagger] = -I$$

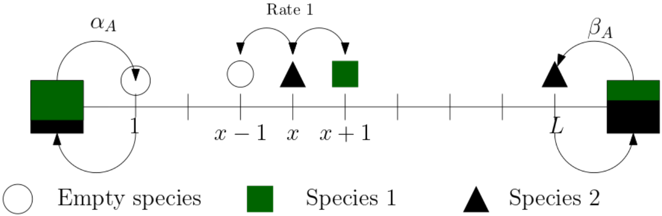
$d(x, n) = x^n$ such that

$$(ad(x,\cdot))(n) = (Ad(\cdot, n))(x) \qquad \left(a^\dagger d(x,\cdot)\right)(n) = \left(A^\dagger d(\cdot, n)\right)(x)$$

$$\mathcal{L}^{\text{dual}} = \frac{1}{2}a^2(I - a^\dagger)a^\dagger \quad \Rightarrow \quad (\mathcal{L}d(\cdot, n))(x) = \left(\mathcal{L}^{\text{dual}}d(x,\cdot)\right)(n)$$

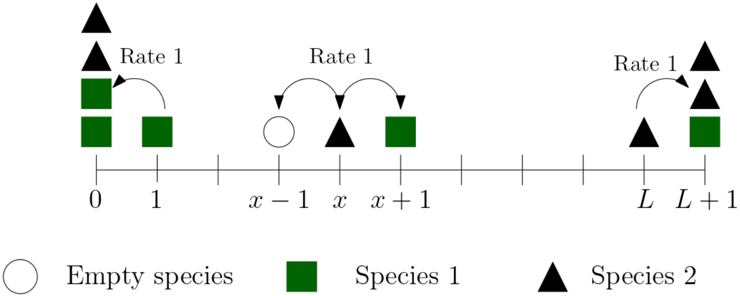
Kingman-Coalesent.

Multi-species stirring process



$$\mathcal{L} = \mathcal{L}_{\text{left}} + \sum_{x=1}^{L-1} \mathcal{L}_{x,x+1} + \mathcal{L}_{\text{right}}$$

Absorbing dual process



$$\mathcal{L}^{\text{dual}} = \mathcal{L}^{\text{dual}}_{\text{left}} + \sum_{x=1}^{L-1} \mathcal{L}_{x,x+1} + \mathcal{L}^{\text{dual}}_{\text{right}}$$

Algebraic description

- Lie algebra $gl(N)$
- Representation $\lambda = (1, 0, \dots, 0)$
- Space $\mathbb{C}^N$: $|\eta_1, \dots, \eta_N\rangle$, $\eta_A \in \{0, 1\}$ and $\eta_N = 1 - \sum_{a=1}^{N-1} \eta_a$
- Algebra generator
  $E_{A,B}|\eta_1, \dots, \eta_N\rangle = \eta_B|\eta_1, \dots, \eta_A + 1, \dots, \eta_B - 1, \dots, \eta_N\rangle$

Bulk generator (second Casimir!)

$$\mathcal{L}_{x,x+1} = \sum_{A,B=1}^{N} \left(E_{A,B}^{[x]} E_{B,A}^{[x+1]} - E_{B,B}^{[x]} E_{A,A}^{[x+1]} \right)$$

Boundary generator (left)

$$\mathcal{L}_{\text{left}} = \sum_{A,B=1}^{N} \alpha_A \left(E_{A,B}^{[1]} - E_{B,B}^{[1]} \right)$$

Duality relation

$$\mathcal{L}^T D = D \mathcal{L}^{\text{dual}} \,.$$

Self-duality for the bulk

- take $E_{A,B} \Rightarrow E_{B,A}$ is conjugate
- $d = \mathrm{Id} \Rightarrow E_{A,B}^T d = dE_{B,A}$
- Dual generator

$$\begin{aligned}\mathcal{L}_{x,x+1}^{\text{dual}} =& \sum_{A,B} E_{B,A}^{[x]} E_{A,B}^{[x+1]} - E_{A,A}^{[x]} E_{B,B}^{[x+1]} \\ =& \mathcal{L}_{x,x+1}\end{aligned}$$

- $\mathcal{L}_{x,x+1}$ central $\Rightarrow D_x = \exp\left\{\sum_{a=1}^{N-1} E_{a,N}^{[x]}\right\} d$

satisfy

$$\mathcal{L}_{x,x+1}^T D_x D_{x+1} = D_x D_{x+1} \mathcal{L}_{x,x+1}^{\text{dual}}$$

Self duality with $D_{\text{bulk}} = \bigotimes_{x=1}^{L} D_x$

Absorbing dual boundaries

$$D_1^{-1}\mathcal{L}_{\text{left}}^T D_1 = \sum_{a=1}^{N-1}\left(\alpha_a E_{N,a}^{[1]} - E_{a,a}^{[1]}\right)$$

Add extra-site bosonic operator and construct

$$\mathcal{L}_{\text{left}}^{\text{dual}} = \sum_{a=1}^{N}\left(\left(\mathbf{a}_a^{[0]}\right)^\dagger E_{N,a}^{[1]} - E_{a,a}^{[1]}\right)$$

define extra-site intertwiner $\mathcal{D}_0$ such that

$$\mathcal{L}_{\text{left}}^T \mathcal{D}_0 D_1 = \mathcal{D}_0 D_1 \mathcal{L}_{\text{left}}^{\text{dual}}$$

Duality matrix

$$D = \mathcal{D}_0 \otimes D_{\text{bulk}} \otimes \mathcal{D}_{L+1}$$

Exact solution

Integrable model (XXX-quantum chain, extra Yang-Baxter symmetry).

- Duality: triangularize the generator
- Matrix product ansatz (MPA) for the non-equilibrium steady state, i.e. solution to $H|\Psi\rangle = 0$
- Combining duality with MPA $\Rightarrow$ exact formulas for $|\Psi\rangle$

Currently: non-compact processes

Boundary driven multi-species harmonic process and integrable heat conduction process (IHC)

- Infinite dimensional representations of $gl(N)$
- Non-trivial symmetries: H is not the Casimir, but a non-linear function of it.
- Self duality harmonic.
- Duality between harmonic and IHC
- Exact solvability:
  - Physics: Yang-Baxter symmetries, Bethe ansatz solutions
  - Markov processes: Hidden parameter model, propagation of invariant measures
- Kardar-Parisi-Zhang?

Summary and open questions

Summary

- Markov duality: study observable via a dual (simpler) process.
- Duality produced via Lie algebra reps
- Contributions:
  - Combine duality and integrable systems to non-equilibrium steady state
  - Extend results to colored processes

Open question?

- Asymmetric non-compact boundary driven processes
- Mapping non-eq onto eq

Thank you for your attention

Appendix

$$D(\boldsymbol{n},\boldsymbol{\xi}) = \left(\prod_{a=1}^{N-1}\alpha_a^{\xi_a^0}\right)\left(\prod_{x=1}^{L}\prod_{a=1}^{N-1}\mathbb{1}_{\{n_a^x\geqslant\xi_a^x\}}\right)\left(\prod_{a=1}^{N-1}\beta_a^{\xi_a^{L+1}}\right)$$

$$|\Psi\rangle = \frac{1}{Z_L}\langle\langle W|\begin{pmatrix}X_1\\ \vdots\\ X_N\end{pmatrix}\otimes\ldots\otimes\begin{pmatrix}X_1\\ \vdots\\ X_N\end{pmatrix}|V\rangle\rangle$$

$\lambda = (\nu, 0, \ldots, 0)$

$$D(\boldsymbol{n},\boldsymbol{\xi}) = \left(\prod_{a=1}^{N-1}(\alpha_a^x)^{\xi_a^0}\right)\prod_{x=1}^{L}\left(\frac{(\nu-\sum_{a=1}^{N-1}\xi_a^x)!}{\nu!}\prod_{a=1}^{N-1}\frac{n_a^x!}{(n_a^x-\xi_a^x)!}\right)\left(\prod_{a=1}^{N-1}(\beta_a^x)^{\xi_a^{L+1}}\right)$$