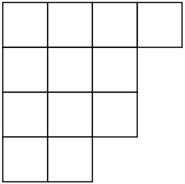
Affine Grassmannian elements, Macdonald identities and hook length formulae

ANR CORTIPOM Closing conference
Le Croisic
9-13 June, 2025

Cédric Lecouvey, David Wahiche
ANR CORTIPOM, Université de Tours – Institut Denis Poisson
10/06/2025

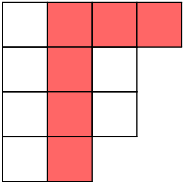
Ferrers diagram and hooks of partitions



$$|\lambda| = 4 + 3 + 3 + 2 = 12$$

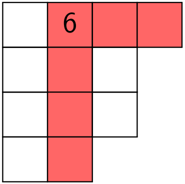
$$(4, 3, 3, 2) \in \mathcal{P}$$

Ferrers diagram and hooks of partitions



$(4,3,3,2) \in \mathcal{P}$

Ferrers diagram and hooks of partitions



$(4, 3, 3, 2) \in \mathcal{P}$

Ferrers diagram and hooks of partitions

7	6	4	1
5	4	2	
4	3	1	
2	1		

$(4,3,3,2) \in \mathcal{P}$

0	1	2	3
-1	0	1	
-2	-1	0	
-3	-2		

contents of $(4,3,3,2) \in \mathcal{P}$

- $\mathcal{H}(\lambda) :=$ {hook-lengths}

Ferrers diagram and hooks of partitions

7	6	4	1
5	4	2	
4	3	1	
2	1	$\mathcal{H}_3$	

$(4, 3, 3, 2) \in \mathcal{P}$

- $\mathcal{H}(\lambda) := \{\text{hook-lengths}\}$
- for $n \in \mathbb{N}^*$, $\mathcal{H}_{\mathbf{n}}(\lambda) := \{h \in \mathcal{H}(\lambda) \mid h \equiv 0 \pmod{n}\}$

Ferrers diagram and hooks of partitions

7	6	4	1
5	4	2	
4	3	1	
2	1	$\mathcal{H}_3$	

$(4, 3, 3, 2) \in \mathcal{P}$

0	1	2	0
2	0	1	
1	2	0	
0	1		

residues mod 3 of $(4, 3, 3, 2) \in \mathcal{P}$

- $\mathcal{H}(\lambda) :=$ {hook-lengths}
- for $n \in \mathbb{N}^*$, $\mathcal{H}_{\mathbf{n}}(\lambda) := \{h \in \mathcal{H}(\lambda) \mid h \equiv 0 \pmod{n}\}$
- $\lambda \in \mathcal{P}_{(n)} \iff \mathcal{H}_n(\lambda) = \emptyset$

The Nekrasov–Okounkov formula in type A

Nekrasov–Okounkov (2006), Westbury (2006), Han (2008)

$$\sum_{\lambda \in \mathcal{P}} q^{|\lambda|} \prod_{h \in \mathcal{H}(\lambda)} \left(\frac{h-z}{h} \right) \left(\frac{h+z}{h} \right) = (q;q)_\infty^{z^2-1}$$

where $(a;q)_\infty := (1-a)(1-aq)(1-aq^2)\cdots$

The Nekrasov–Okounkov formula in type A

Nekrasov–Okounkov (2006), Westbury (2006), Han (2008)

$$\sum_{\lambda \in \mathcal{P}} q^{|\lambda|} \prod_{h \in \mathcal{H}(\lambda)} \left(\frac{h-z}{h} \right) \left(\frac{h+z}{h} \right) = (q;q)_\infty^{z^2-1}$$

where $(a;q)_\infty := (1-a)(1-aq)(1-aq^2)\cdots$

Note $z = n \in \mathbb{N}^* \Rightarrow \lambda$ are n-cores.

The Nekrasov–Okounkov formula in type A

Nekrasov–Okounkov (2006), Westbury (2006), Han (2008)

$$\sum_{\lambda \in \mathcal{P}} q^{|\lambda|} \prod_{h \in \mathcal{H}(\lambda)} \left(\frac{h-z}{h}\right)\left(\frac{h+z}{h}\right) = (q;q)_\infty^{z^2-1}$$

where $(a;q)_\infty := (1-a)(1-aq)(1-aq^2)\cdots$

Note $z = n \in \mathbb{N}^* \Rightarrow \lambda$ are n-cores.

Outline of Han's proof:

- specialization of Macdonald identity in type $A_{n-1}^{(1)}$ for any odd n
- lift the equality to any complex z by a polynomiality argument

Weyl denominator formula

For $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}^n$, set $x^\beta = x_1^{\beta_1} \ldots x_n^{\beta_n}$

$$\prod_{1 \leq i < j \leq n} (x_i - x_j) = \det(x_i^{n-j}) = \sum_{\sigma \in S_n} \varepsilon(\sigma) x^{\sigma(\rho)}$$

with $\rho = (n-1, n-2, \ldots, 1, 0)$

Weyl denominator formula

For $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{N}^n$, set $x^\beta = x_1^{\beta_1} \dots x_n^{\beta_n}$

$$\prod_{1 \leq i < j \leq n} (x_i - x_j) = \det(x_i^{n-j}) = \sum_{\sigma \in S_n} \varepsilon(\sigma) x^{\sigma(\rho)}$$

with $\rho = (n-1, n-2, \dots, 1, 0)$
$E = \bigoplus_{i=1}^n \mathbb{R}\varepsilon_i$
S_n : subgroup of $\mathrm{GL}(E)$ generated by reflections $s_i, i = 1, \dots, n-1$ through the hyperplanes $(\mathbb{R}\alpha_i)^\perp$ where $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$.

Weyl denominator formula

For $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}^n$, set $x^\beta = x_1^{\beta_1} \ldots x_n^{\beta_n}$

$$\prod_{1 \leq i < j \leq n} (x_i - x_j) = \det(x_i^{n-j}) = \sum_{\sigma \in S_n} \varepsilon(\sigma) x^{\sigma(\rho)}$$

with $\rho = (n-1, n-2, \ldots, 1, 0)$
$E = \bigoplus_{i=1}^n \mathbb{R}\varepsilon_i$
S_n : subgroup of $\mathrm{GL}(E)$ generated by reflections $s_i, i = 1, \ldots, n-1$ through the hyperplanes $(\mathbb{R}\alpha_i)^\perp$ where $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$.

- $s_i^2 = 1$
- $s_i s_j = s_j s_i, |i-j| > 1$
- $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$

Weyl denominator formula

For $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{N}^n$, set $x^\beta = x_1^{\beta_1} \dots x_n^{\beta_n}$

$$\prod_{1 \le i < j \le n} (x_i - x_j) = \det(x_i^{n-j}) = \sum_{\sigma \in S_n} \varepsilon(\sigma) x^{\sigma(\rho)}$$

with $\rho = (n-1, n-2, \dots, 1, 0)$
$E = \bigoplus_{i=1}^n \mathbb{R}\varepsilon_i$
S_n : subgroup of $\mathrm{GL}(E)$ generated by reflections
$s_i, i = 1, \dots, n-1$ through the hyperplanes $(\mathbb{R}\alpha_i)^\perp$ where
$\alpha_i = \varepsilon_i - \varepsilon_{i+1}$.
Reflections in S_n are those of hyperplanes $(\mathbb{R}\alpha)^\perp$ where
$\alpha = \varepsilon_i - \varepsilon_j$ and $1 \le i < j \le n$.

Weyl denominator formula

For $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{N}^n$, set $x^\beta = x_1^{\beta_1} \dots x_n^{\beta_n}$

$$\prod_{1\leq i<j\leq n} (x_i - x_j) = \det(x_i^{n-j}) = \sum_{\sigma\in S_n} \varepsilon(\sigma) x^{\sigma(\rho)}$$

with $\rho = (n-1, n-2, \dots, 1, 0)$
$E = \bigoplus_{i=1}^n \mathbb{R}\varepsilon_i$
S_n : subgroup of GL(E) generated by reflections $s_i, i = 1, \dots, n-1$ through the hyperplanes $(\mathbb{R}\alpha_i)^\perp$ where $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$.
Reflections in S_n are those of hyperplanes $(\mathbb{R}\alpha)^\perp$ where $\alpha = \varepsilon_i - \varepsilon_j$ and $1 \leq i < j \leq n$.
$s_{\alpha_1}(x_1, x_2, x_3) = (x_2, x_1, x_3)$

Weyl denominator formula

For $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{N}^n$, set $x^\beta = x_1^{\beta_1} \dots x_n^{\beta_n}$

$$\prod_{1 \le i < j \le n} (x_i - x_j) = \det(x_i^{n-j}) = \sum_{\sigma \in S_n} \varepsilon(\sigma) x^{\sigma(\rho)}$$

with $\rho = (n-1, n-2, \dots, 1, 0)$
$E = \bigoplus_{i=1}^n \mathbb{R}\varepsilon_i$
S_n : subgroup of $\mathrm{GL}(E)$ generated by reflections $s_i, i = 1, \dots, n-1$ through the hyperplanes $(\mathbb{R}\alpha_i)^\perp$ where $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$.
Reflections in S_n are those of hyperplanes $(\mathbb{R}\alpha)^\perp$ where $\alpha = \varepsilon_i - \varepsilon_j$ and $1 \le i < j \le n$.
Set $R^+ = \{\varepsilon_i - \varepsilon_j, 1 \le i < j \le n\}$

$$\prod_{1 \le i < j \le n} \left(1 - \frac{x_i}{x_j}\right) = \prod_{\alpha \in R^+} (1 - e^{-\alpha}) = \sum_{\sigma \in S_n} \varepsilon(\sigma) x^{\sigma(\rho) - \rho}$$

Macdonald identity

Macdonald (1972): analogues of Weyl denominator formula for affine root systems
reformulation by Stanton (1989), Rosengren–Schlosser (2006) for the 7 infinite affine types

Macdonald identity

Macdonald (1972): analogues of Weyl denominator formula for affine root systems
reformulation by Stanton (1989), Rosengren–Schlosser (2006) for the 7 infinite affine types
In particular in type $A_{n-1}^{(1)}$: $x_i = e^{-\varepsilon_i}$,$q = e^{-\delta}$

Macdonald identity

Macdonald (1972): analogues of Weyl denominator formula for affine root systems
reformulation by Stanton (1989), Rosengren–Schlosser (2006) for the 7 infinite affine types
In particular in type $A_{n-1}^{(1)}$: $x_i = e^{-\varepsilon_i}, q = e^{-\delta}$

$$\sum_{\substack{\mathbf{m}\in\mathbb{Z}^n \\ m_1+\cdots+m_n=0}} q^{n\|m\|^2/2+\sum_{i=1}^{n}(i-1)m_i} \det_{1\le i,j\le n}\left(x_i^{nm_j+j-1}\right)$$

$$= (q;q)_\infty^{n-1} \prod_{1\le i<j\le n} (x_i/x_j;q)_\infty (qx_j/x_i;q)_\infty$$

Macdonald identity

Macdonald (1972): analogues of Weyl denominator formula for affine root systems
reformulation by Stanton (1989), Rosengren–Schlosser (2006) for the 7 infinite affine types
In particular in type $A_{n-1}^{(1)}$: $x_i = e^{-\varepsilon_i}$,$q = e^{-\delta}$
Divide by positive roots of A_{n-1}: Vandermonde determinant
Garvan–Kim–Stanton (1990):
$\omega \in \mathcal{P}_{(n)} \longleftrightarrow (n_0, \ldots, n_{n-1}) \in \mathbb{Z}^n$ such that $\sum_{i=0}^{n-1} n_i = 0$.

$$\sum_{\omega \in \mathcal{P}_{(n)}} (-1)^{|H_{<n}|} q^{|\omega|} s_\mu(\mathbf{x}) \prod_{i=1}^{n} x_i^{-\ell(\omega)} = (q;q)_\infty^{n-1} \prod_{1 \le i < j \le n} (qx_i/x_j;q)_\infty (qx_j/x_i;q)_\infty$$

Macdonald identity

Macdonald (1972): analogues of Weyl denominator formula for affine root systems
reformulation by Stanton (1989), Rosengren–Schlosser (2006) for the 7 infinite affine types
In particular in type $A_{n-1}^{(1)}$: $x_i = e^{-\varepsilon_i}, q = e^{-\delta}$
Question: what is the uniform object in all types ?

The affine symmetric group $\tilde{S}_n$

$\tilde{S}_n$ is the subgroup of n-periodic permutations generated by

$$s_i(m) = \begin{cases} m+1 \text{ if } m \equiv i \pmod n \\ m-1 \text{ if } m \equiv i+1 \pmod n \\ m \text{ otherwise} \end{cases}$$

The affine symmetric group $\tilde{S}_n$

$\tilde{S}_n$ is the subgroup of n-periodic permutations generated by

$$s_i(m) = \begin{cases} m+1 \text{ if } m \equiv i \pmod{n} \\ m-1 \text{ if } m \equiv i+1 \pmod{n} \\ m \text{ otherwise} \end{cases}$$

$$\begin{aligned} \tilde{S}_n &= \langle s_0, s_1, \ldots, s_{n-1} \rangle \\ s_i^2 &= 1 \\ s_{i+1}s_is_{i+1} &= s_is_{i+1}s_i \\ s_is_j &= s_js_i \text{ for } i-j \not\equiv 0, 1, n-1 \pmod{n} \end{aligned}$$

The affine symmetric group $\tilde{S}_n$

$\tilde{S}_n$ is the subgroup of n-periodic permutations generated by

$$s_i(m) = \begin{cases} m+1 \text{ if } m \equiv i \pmod{n} \\ m-1 \text{ if } m \equiv i+1 \pmod{n} \\ m \text{ otherwise} \end{cases}$$

The reflections in $\tilde{S}_n$ are the elements of type

$$w s_i w^{-1}, w \in \tilde{S}_n, i = 0, \ldots, n-1$$

The affine symmetric group $\tilde{S}_n$

The reflections in $\tilde{S}_n$ are the elements of type

$$ws_i w^{-1}, w \in \tilde{S}_n, i = 0, \dots, n-1$$

$$(ij)_k \text{ with } 1 \leq i \neq j \leq n \text{ with } \begin{cases} k \geq 0 \text{ if } i < j \\ k < 0 \text{ if } i > j \end{cases}$$

such that $(ij)_k(i) = j + kn, (ij)_k(j) = i - kn$

Action of $\tilde{S}_n$ on $\mathcal{P}$

$$\tilde{S}_n = \langle s_0, s_1, \ldots, s_{n-1} \rangle$$
$$s_i^2 = 1$$
$$s_{i+1} s_i s_{i+1} = s_i s_{i+1} s_i$$
$$s_i s_j = s_j s_i \text{ for } i - j \not\equiv 0, 1, n-1 \pmod{n}$$

Lam, Lapointe, Morse,Schilling, Shimozono, Zabrocki

0	1	2	0	1
2	0	1	2	
1	2	0		
0	1	2		
2				

Action of $\tilde{S}_n$ on $\mathcal{P}$

$$\tilde{S}_n = \langle s_0, s_1, \ldots, s_{n-1} \rangle$$
$$s_i^2 = 1$$
$$s_{i+1} s_i s_{i+1} = s_i s_{i+1} s_i$$
$$s_i s_j = s_j s_i \text{ for } i - j \not\equiv 0, 1, n-1 \pmod{n}$$

Lam, Lapointe, Morse,Schilling, Shimozono, Zabrocki

0	1	2	0	1
2	0	1	2	
1	2	0	1	
0	1	2		
2				

$\xrightarrow{s_1}$

0	1	2	0	1
2	0	1	2	
1	2	0		
0	1			
2				

Action of $\tilde{S}_n$ on $\mathcal{P}$

$$\begin{aligned} \tilde{S}_n &= \langle s_0, s_1, \ldots, s_{n-1} \rangle \\ s_i^2 &= 1 \\ s_{i+1} s_i s_{i+1} &= s_i s_{i+1} s_i \\ s_i s_j &= s_j s_i \text{ for } i - j \not\equiv 0, 1, n-1 \pmod{n} \end{aligned}$$

Lam, Lapointe, Morse,Schilling, Shimozono, Zabrocki

$$c \in \tilde{S}_n / S_n \longleftrightarrow \omega \in \mathcal{P}_{(n)} = \tilde{S}_n \cdot \emptyset$$

Affine root system and hook length formulae

Theorem [Dehaye–Han (2011),W.(2025+),Lecouvey-W.(2025)]

Set $n \in \mathbb{N}$ and let τ be a function defined over $\mathbb{Z}$.
Then $\exists\ \phi_n : \omega \in \mathcal{P}_{(n)} \mapsto (v_0, \cdots, v_{n-1})$ V_n-coding such that:

$$|\omega| = \frac{1}{2n} \sum_{i=0}^{n-1} v_i^2 - \frac{n^2-1}{24}$$

Affine root system and hook length formulae

Theorem [Dehaye–Han (2011),W.(2025+),Lecouvey-W.(2025)]

Set $n \in \mathbb{N}$ and let τ be a function defined over $\mathbb{Z}$.
Then $\exists\ \phi_n : \omega \in \mathcal{P}_{(n)} \mapsto (v_0, \cdots, v_{n-1})$ V_n-coding such that:

$$|\omega| = \frac{1}{2n}\sum_{i=0}^{n-1} v_i^2 - \frac{n^2-1}{24}$$

and for $\beta_i(\omega) = \#\{h \in \mathcal{H}(\omega)/h = n-i\}$:

Affine root system and hook length formulae

Theorem [Dehaye–Han (2011),W.(2025+),Lecouvey-W.(2025)]

Set $n \in \mathbb{N}$ and let τ be a function defined over $\mathbb{Z}$.
Then $\exists\ \phi_n : \omega \in \mathcal{P}_{(n)} \mapsto (v_0, \cdots, v_{n-1})$ V_n-coding such that:

$$|\omega| = \frac{1}{2n}\sum_{i=0}^{n-1} v_i^2 - \frac{n^2-1}{24}$$

and for $\beta_i(\omega) = \#\{h \in \mathcal{H}(\omega) / h = n-i\}$:

$$\prod_{h \in \mathcal{H}(\omega)} \frac{\tau(h-n)}{\tau(h)}\frac{\tau(h+n)}{\tau(h)} = \prod_{i=1}^{n-1}\left(\frac{\tau(-i)}{\tau(i)}\right)^{\beta_i(\omega)} \prod_{0 \le i < j \le n-1} \frac{\tau(v_i - v_j)}{\tau(i-j)}.$$

Affine root system and hook length formulae

Theorem [Dehaye–Han (2011),W.(2025+),Lecouvey-W.(2025+)]

Set $n \in \mathbb{N}$ and let τ be a function defined over $\mathbb{Z}$.
Then $\exists\ \phi_n : \omega \in \mathcal{P}_{(n)} \mapsto (v_0, \cdots, v_{n-1})$ V_n-coding such that:

$$|\omega| = \frac{1}{2n} \sum_{i=0}^{n-1} v_i^2 - \frac{n^2-1}{24}$$

and for $\beta_i(\omega) = \#\{h \in \mathcal{H}(\omega) / h = n - i\}$:

$$\prod_{h \in \mathcal{H}(\omega)} \frac{(h-n)}{h} \frac{(h+n)}{h} = \prod_{i=1}^{n-1} \left(\frac{-i}{i}\right)^{\beta_i(\omega)} \prod_{0 \leq i < j \leq n-1} \frac{v_i - v_j}{i - j}.$$

A uniform Nekrasov–Okounkov formula

Uniform formulation for all types :

$$\sum_{\lambda \in \mathcal{S}} q^{|\lambda|} \prod_{h \in \mathcal{H}_X(\lambda)} \frac{(h-z)(h+z)}{h^2} = \prod_{\alpha \in R_a^+ \setminus R^+} (1 - e^{-\alpha})$$

A uniform Nekrasov–Okounkov formula

Uniform formulation for all types :

$$\sum_{\lambda\in\mathcal{S}} q^{|\lambda|} \prod_{h\in\mathcal{H}_X(\lambda)} \frac{(h-z)(h+z)}{h^2} = \prod_{\alpha\in R_a^+\setminus R^+} (1-e^{-\alpha})$$

Thank you !