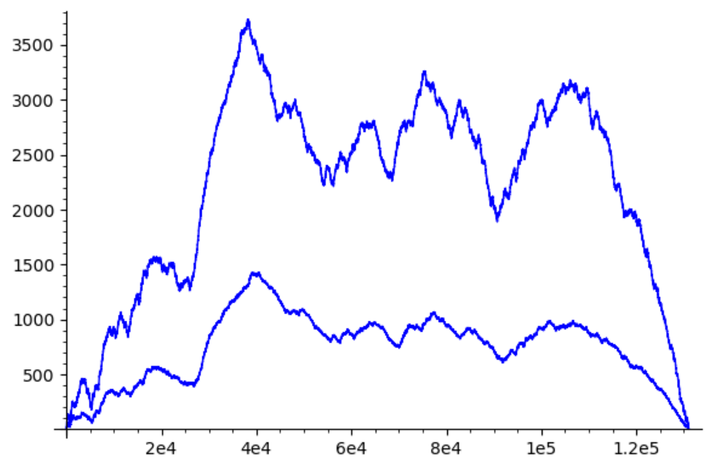
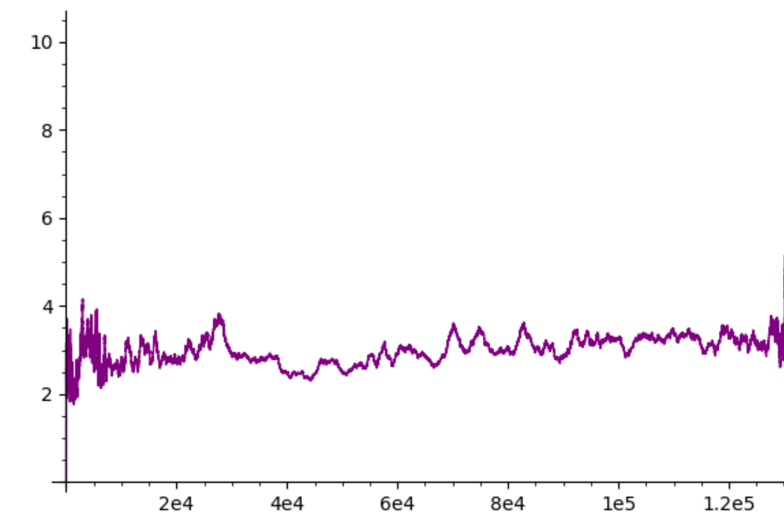
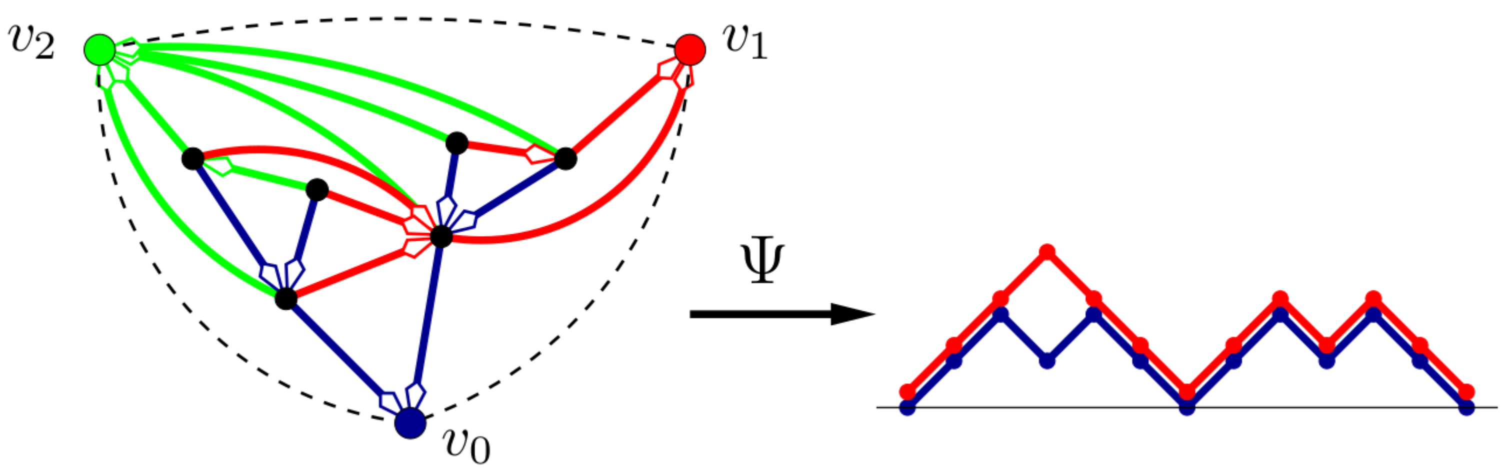
On the scaling of random Tamari intervals and Schnyder woods of random triangulations (with an asymptotic D-finite trick)



Guillaume Chapuy
(CNRS – IRIF – Université Paris Cité)





closing conference of the Cortipom project, le Croisic, June 2025

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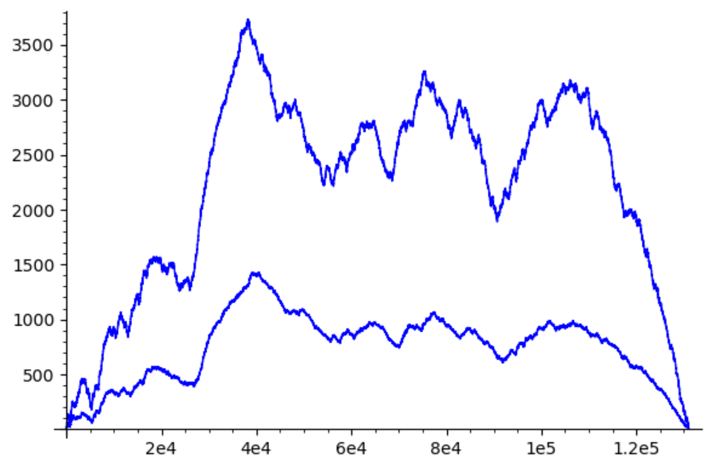

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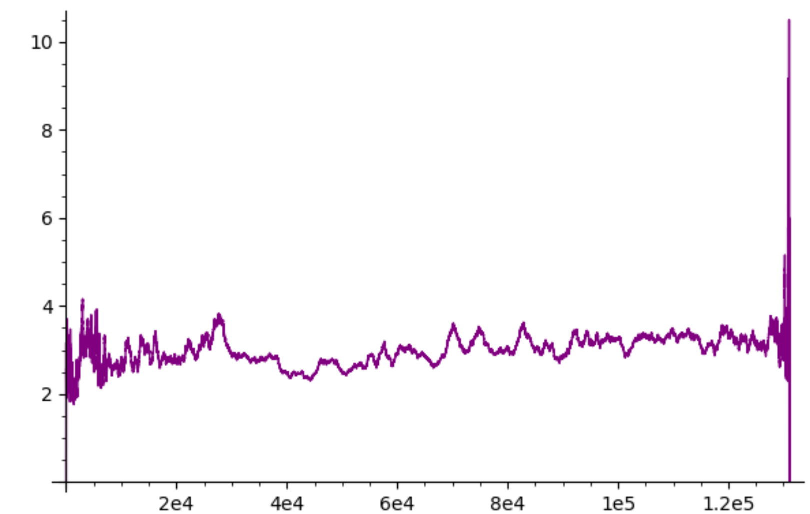

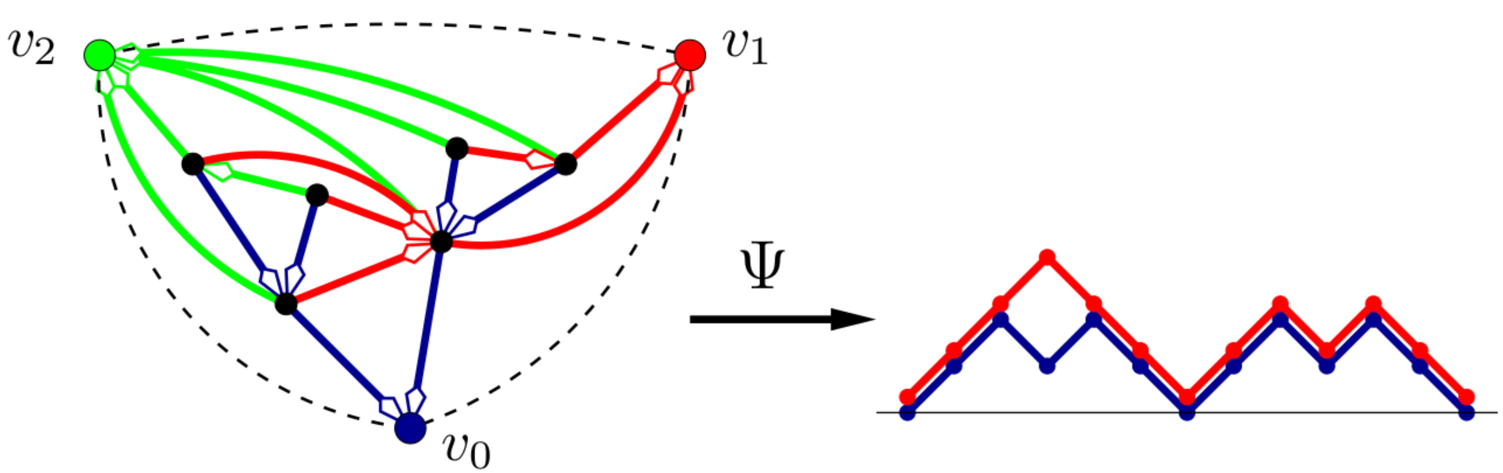

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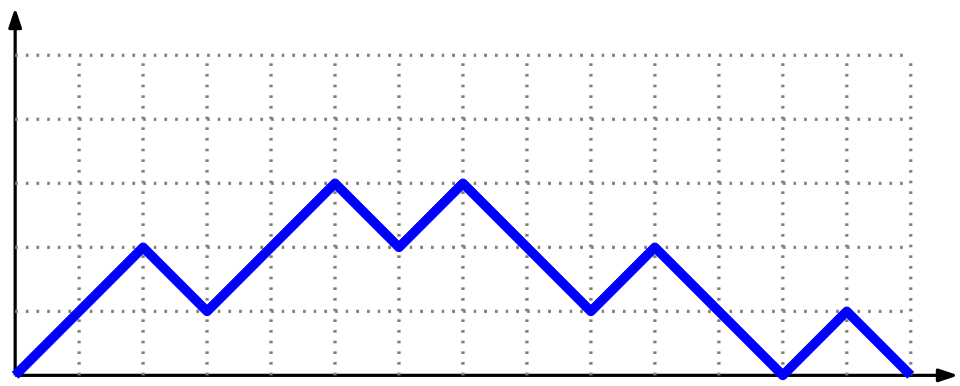
Dyck paths and Tamari partial order

- Dyck path: steps ± 1, goes from 0 to 0, stays ≥ 0, length: $2n$.



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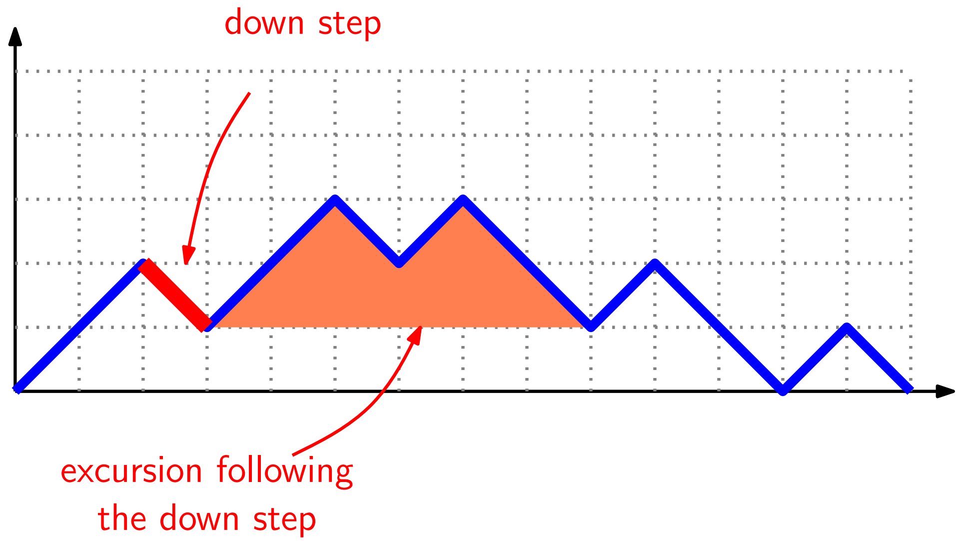
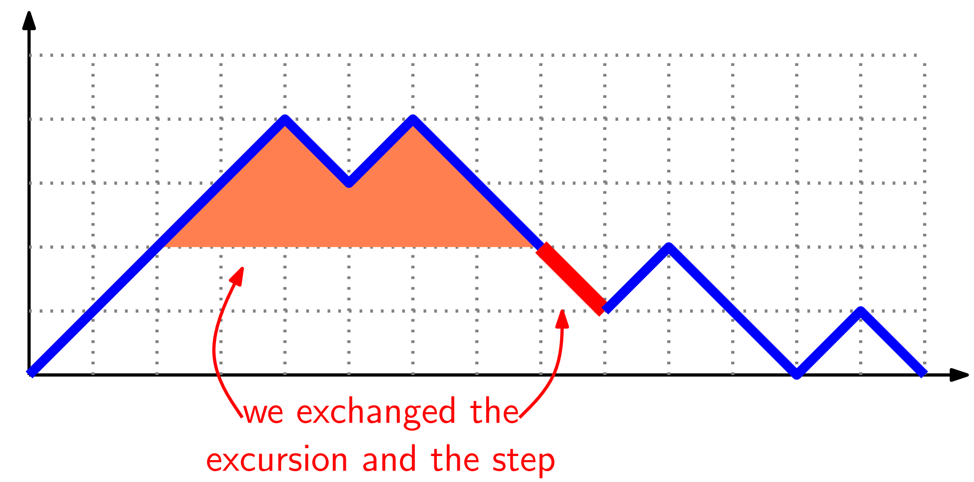


$\preccurlyeq$



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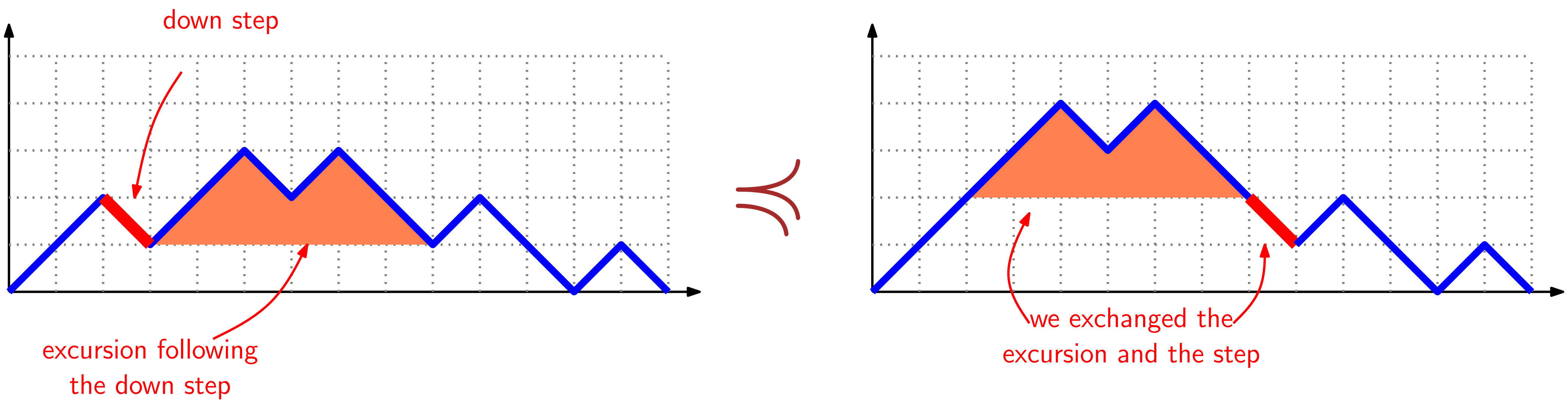
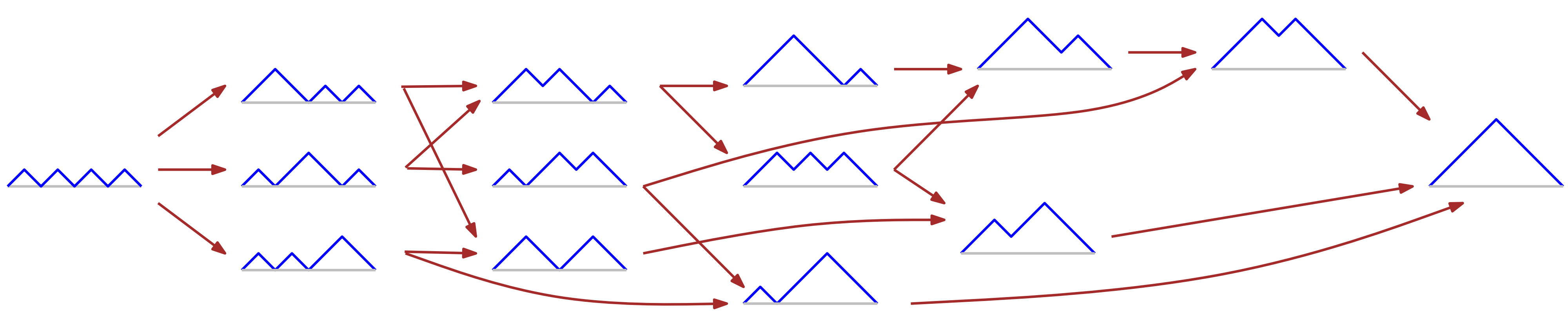


- This is the famous Tamari poset



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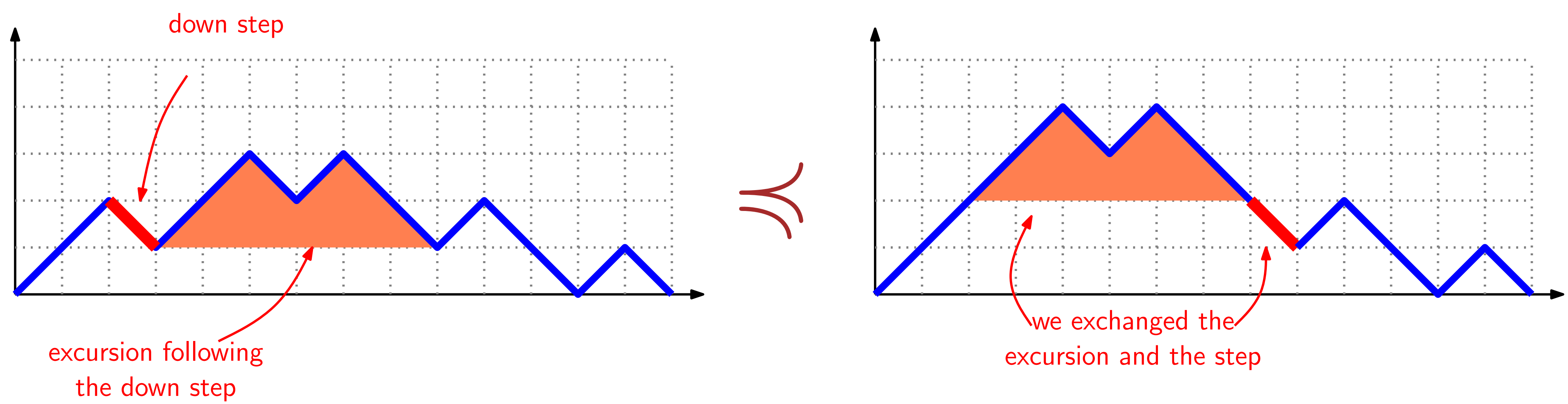

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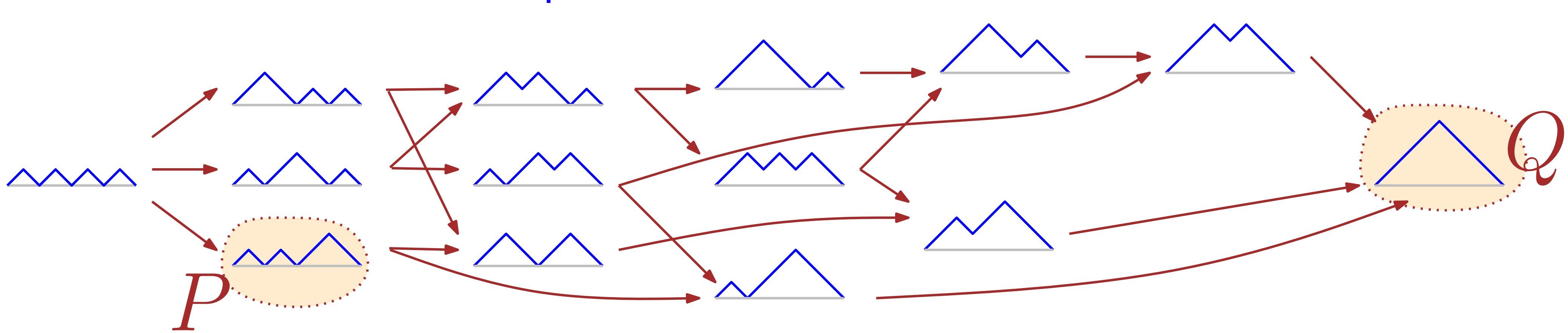

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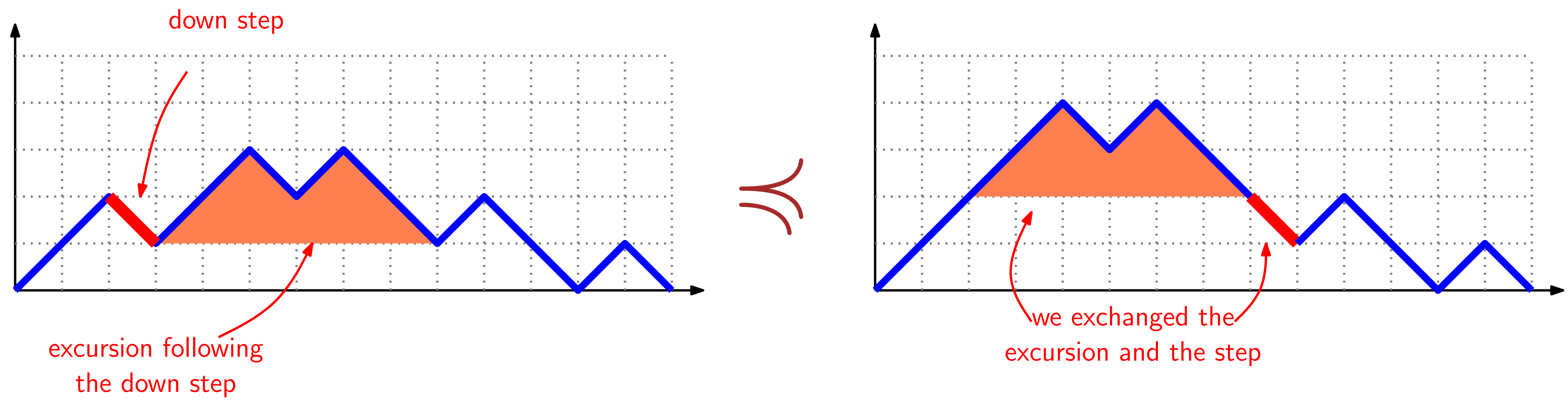

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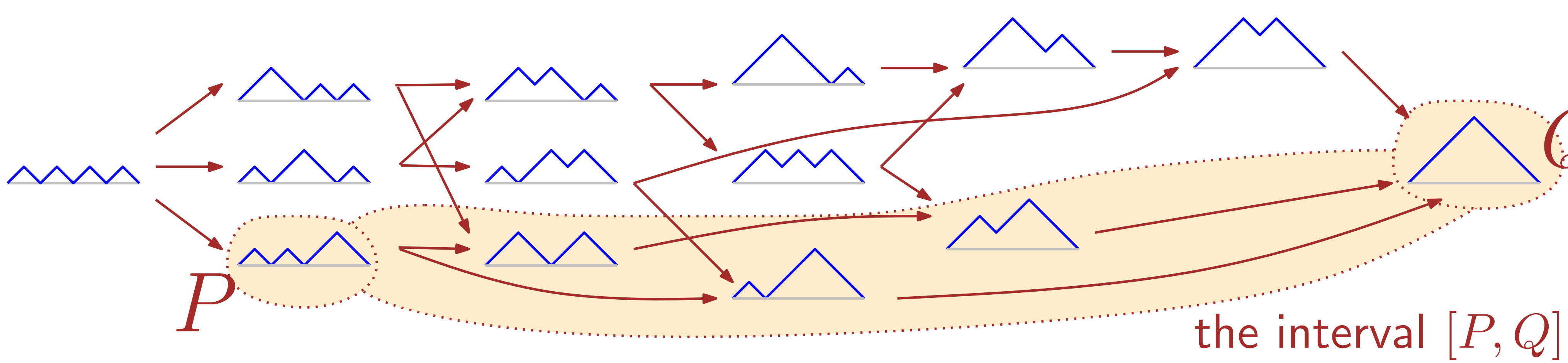

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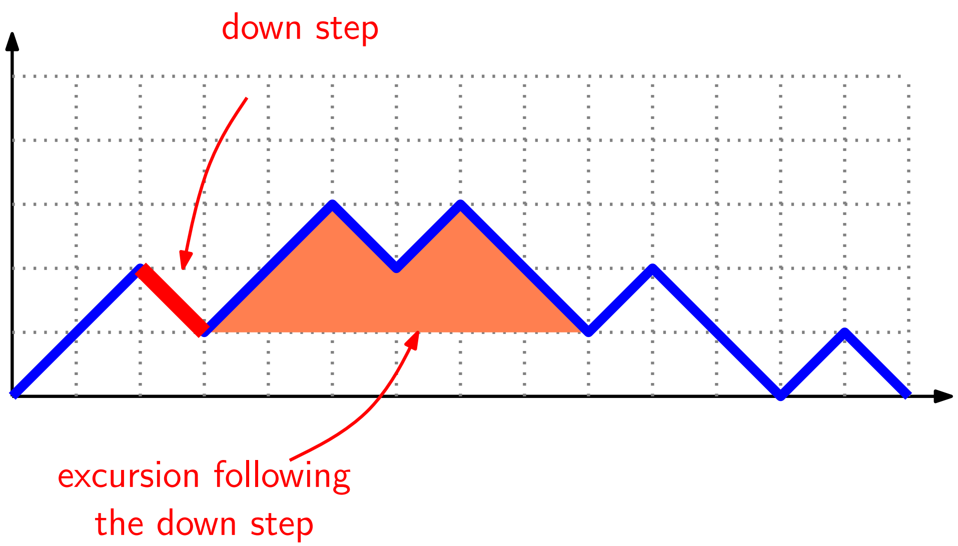

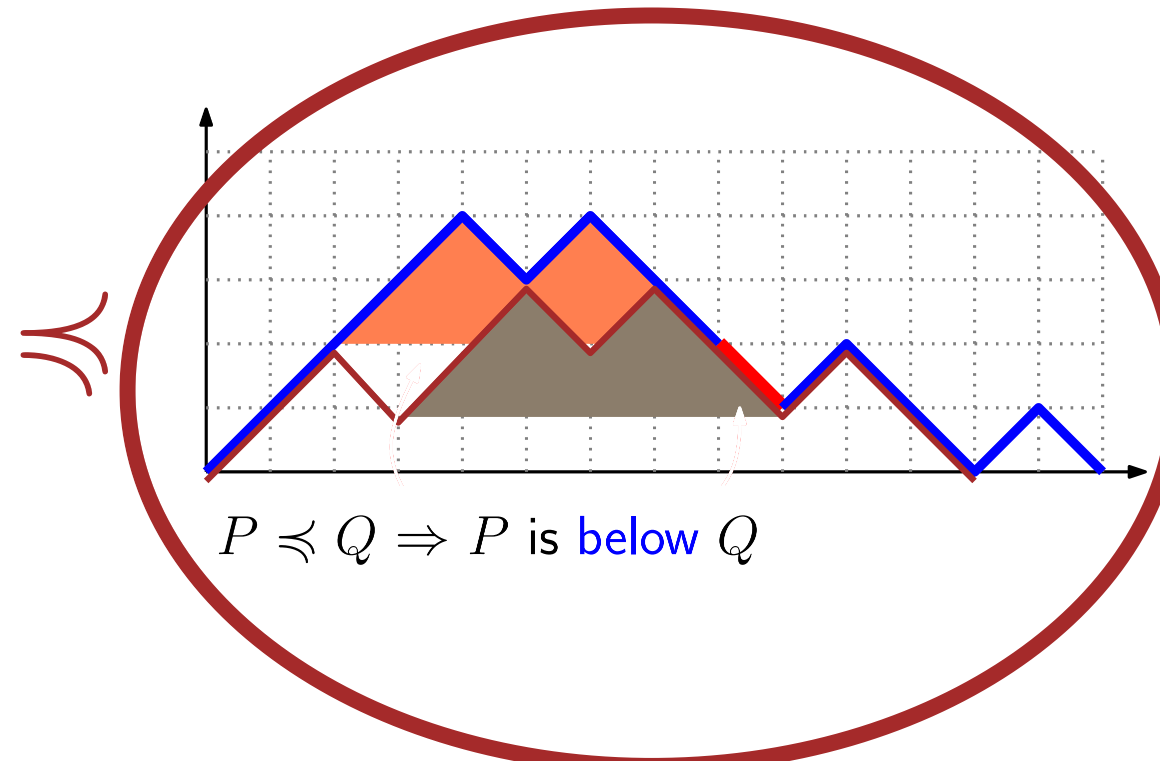

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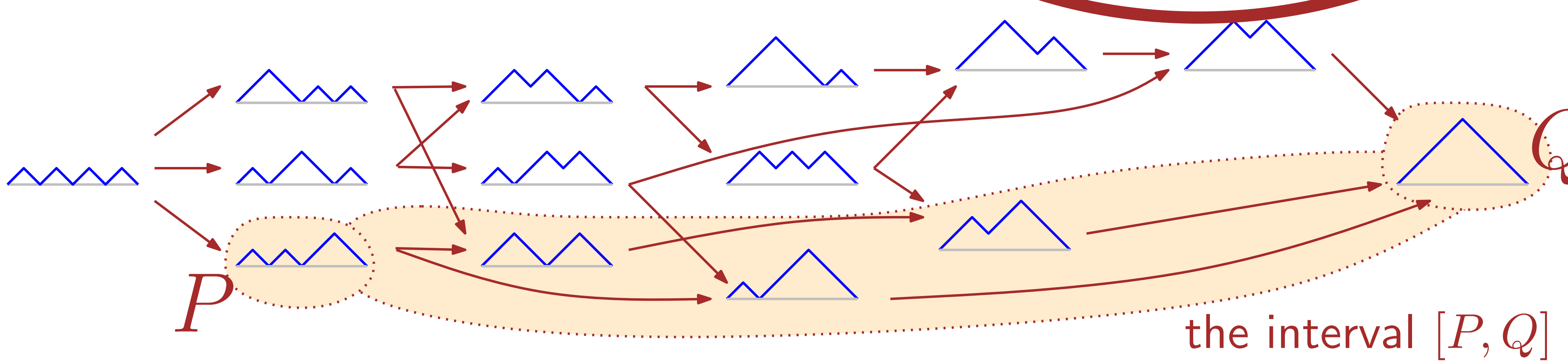
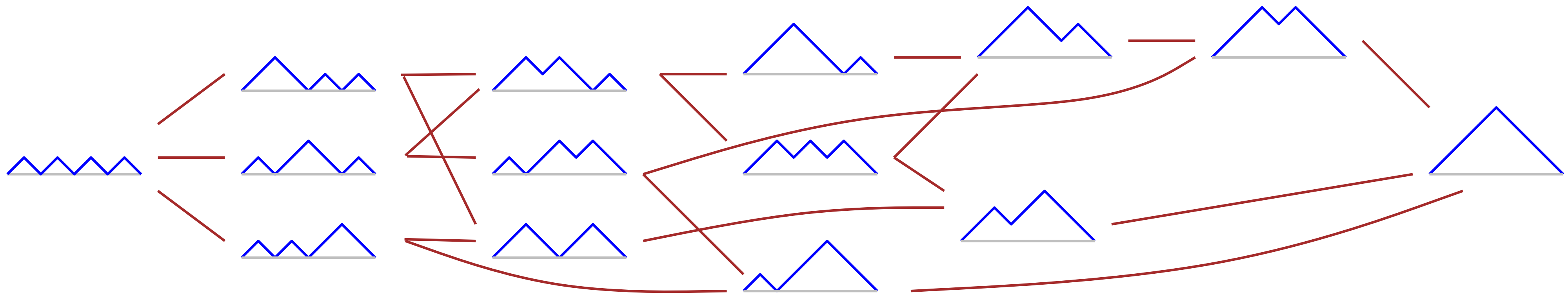


Dyck paths and Tamari partial order - (parenthesis)



- Graph with $\mathrm{Cat}(n)$ vertices

- The mixing time of the simple random walk is unknown! (conjecture $O(n^{3/2})$ Aldous 1990's, impressive partial results by Eppstein+Frishberg)

- The diameter is $2n - o(n)$ [Sleator-Thurston-Tarjan, Pournin], mysterious connections with hyperbolic geometry!

Enumeration of intervals

- [Chapoton 06] The number of pairs $[P, Q]$ such that $P \preccurlyeq Q$ is:

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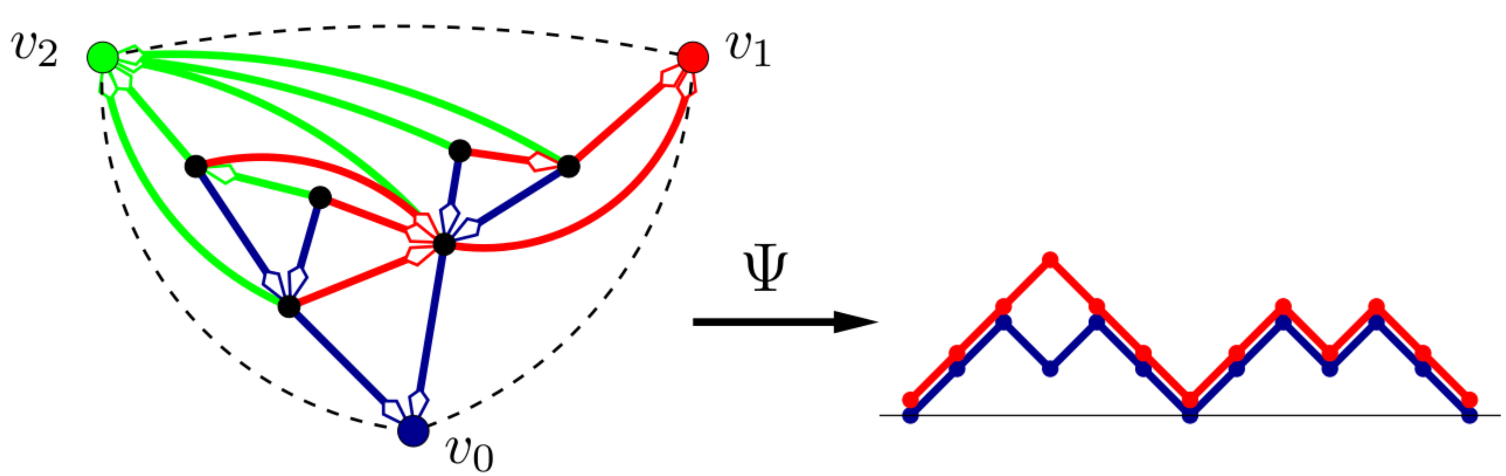

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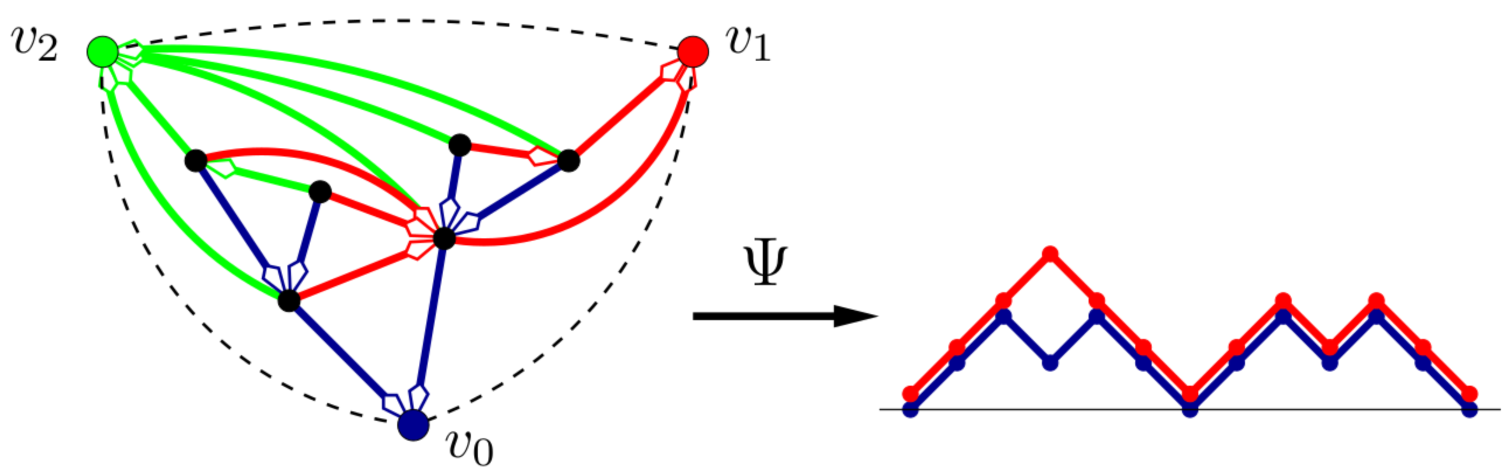


- This is only the beginning of rich analogies between Tamari intervals and maps... see the works of [Fang] and collaborators. And also [MBM-GC-LFPR]

$$(n+1)^{l-2} \prod_{i=1}^{\ell(\lambda)} \binom{2\lambda_i}{\lambda_i} \quad \text{vs} \quad 2(n-1)_{\ell(\lambda)-2} \prod_{i=1}^{\ell(\lambda)} \binom{2\lambda_i - 1}{\lambda_i}$$

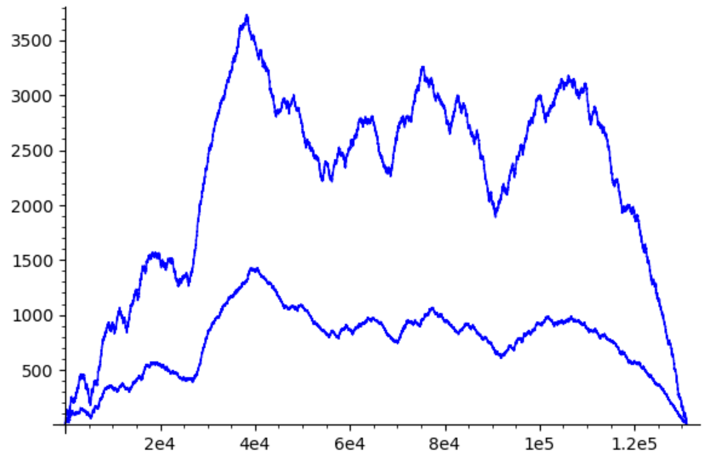
Random Tamari intervals (?)

• **Theorem [C'24].** Let (P_n, Q_n) be a random Tamari interval chosen uniformly in $\mathcal{I}_n$. Let $I \in [0, 2n]$ be a uniformly chosen abscissa. Then:

$$\frac{Q_n(I)}{n^{3/4}} \longrightarrow Z \quad , \quad \mathbf{E}[Z^k] = \frac{\sqrt{3}\cdot 2^{-\frac{k}{4}-1}}{\sqrt{\pi}} \frac{\Gamma(\frac{1}{4}k+\frac{1}{3})\Gamma(\frac{1}{4}k+\frac{2}{3})}{\Gamma(\frac{1}{4}k+\frac{1}{2})}.$$

Note: $Z = (XY)^{1/4}$ where $X \sim \beta(\frac{1}{3}, \frac{1}{6})$ and $Y \sim \Gamma(\frac{2}{3}, \frac{1}{2})$.

paths



Simulations: Wenjie Fang.

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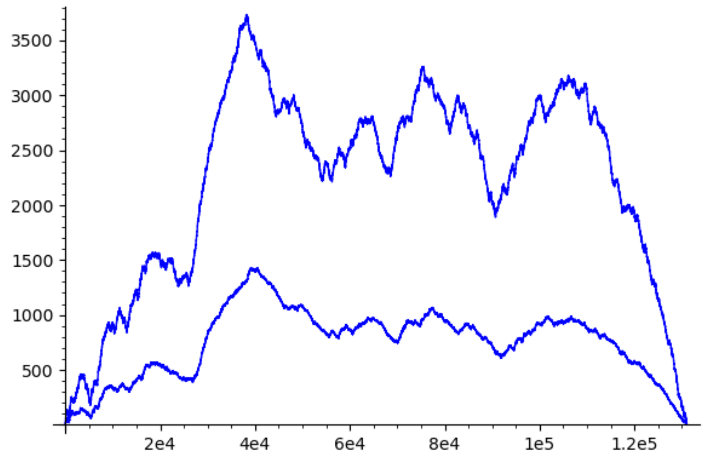

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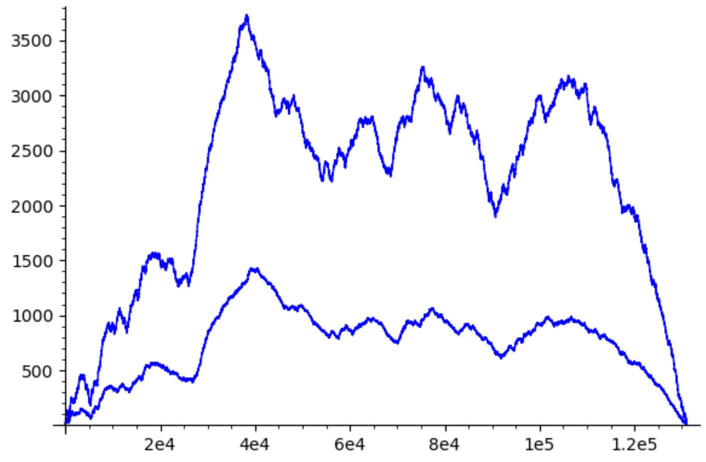

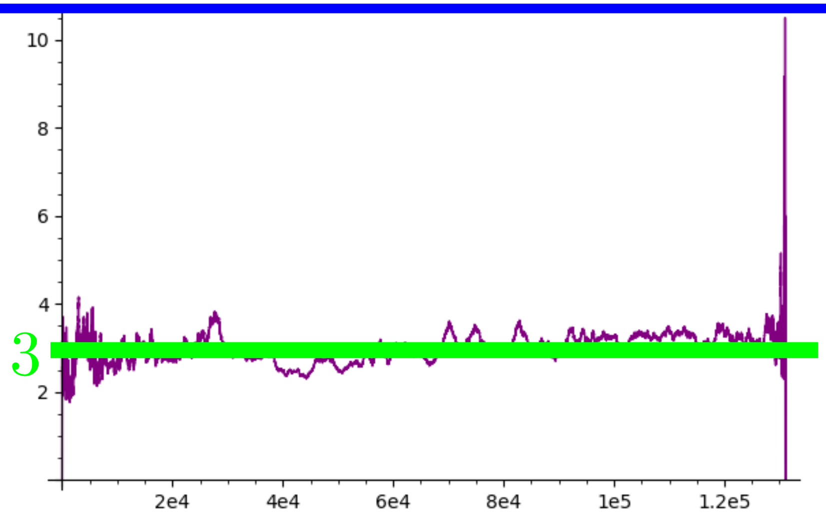

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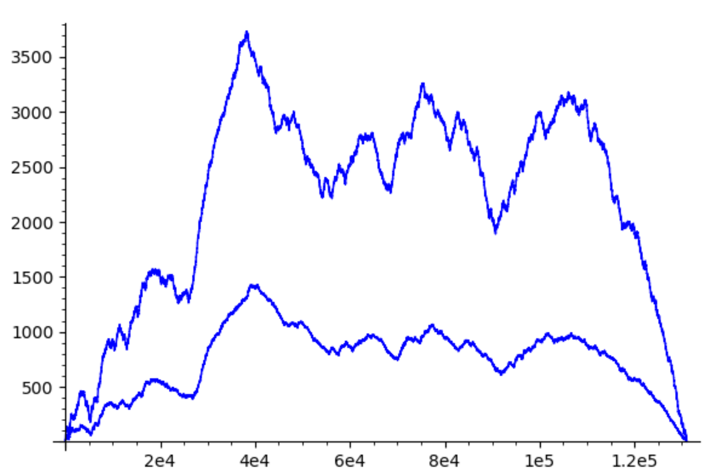

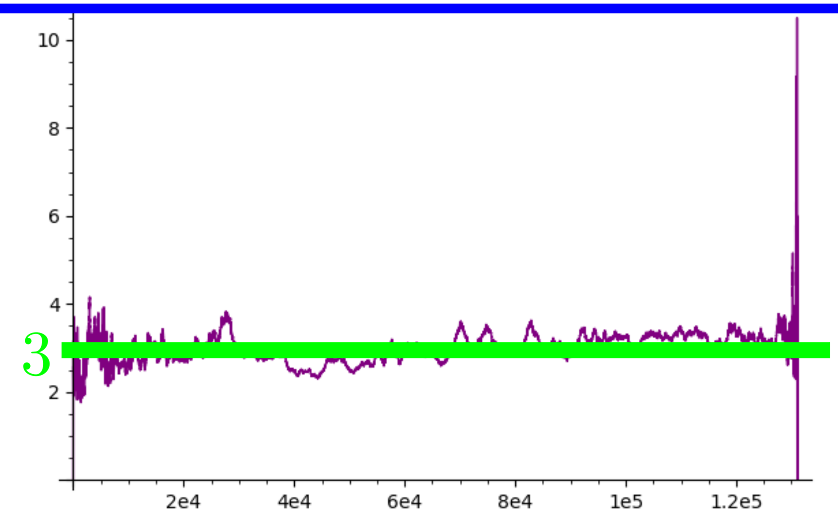

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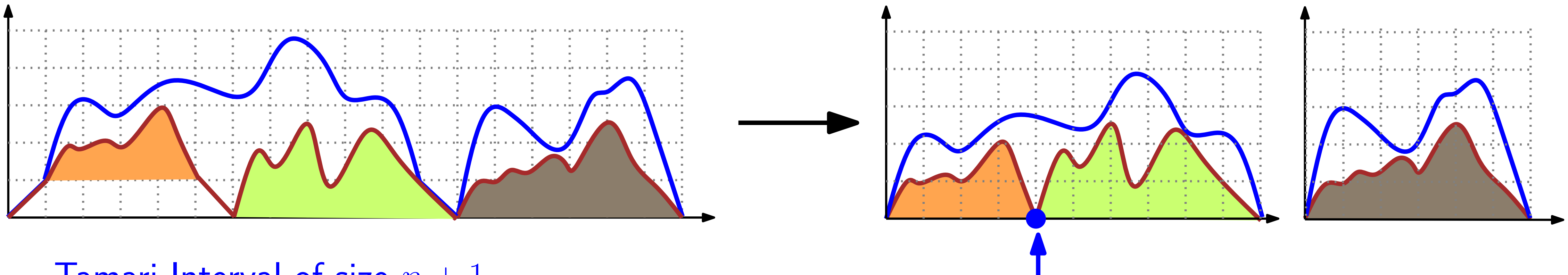
(Classical) Enumeration of Tamari intervals

[[MBM + Éric Fusy + LFPR]]



Tamari Interval of size $n+1$

Tamari Interval with a **marked zero on lower path**

Tamari Interval

total size n

Series of intervals: $F(t;x)$

t: size

x: number of zeroes of the lower path

$$F(t;x) =: \sum_{i\geq 0} F_i(t) x^i$$

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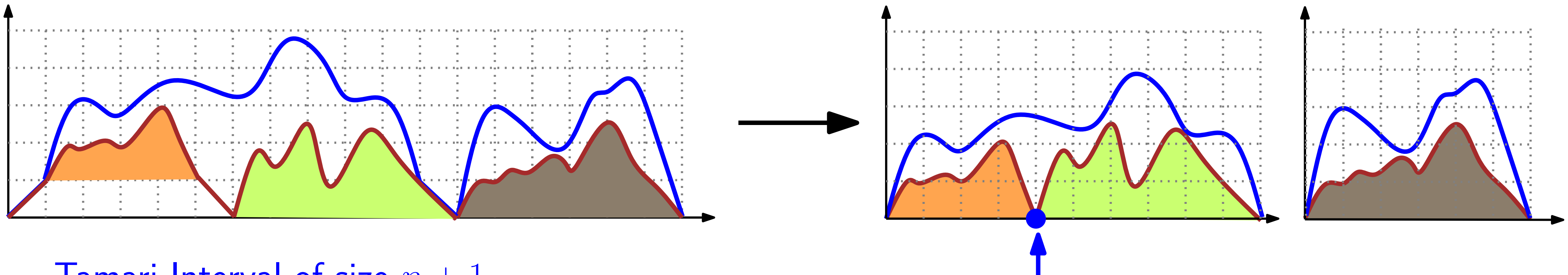

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$$F(t;x) = x + t \sum_{i\geq 1} F_i(t)\left(x + x^2 + \cdots + x^i\right) F(t,x)$$

$$= x + tx \sum_{i\geq 1} F_i(t) \frac{x^i - 1}{x-1} F(t,x)$$

$$= x + tx \frac{F(t,x) - F(t,1)}{x-1} F(t,x)$$

- Polynomial equation with one catalytic variable. Effective theory from Bousquet-Mélou–Jehanne, the solution is algebraic and explicit → Chapoton's theorem on enumeration.

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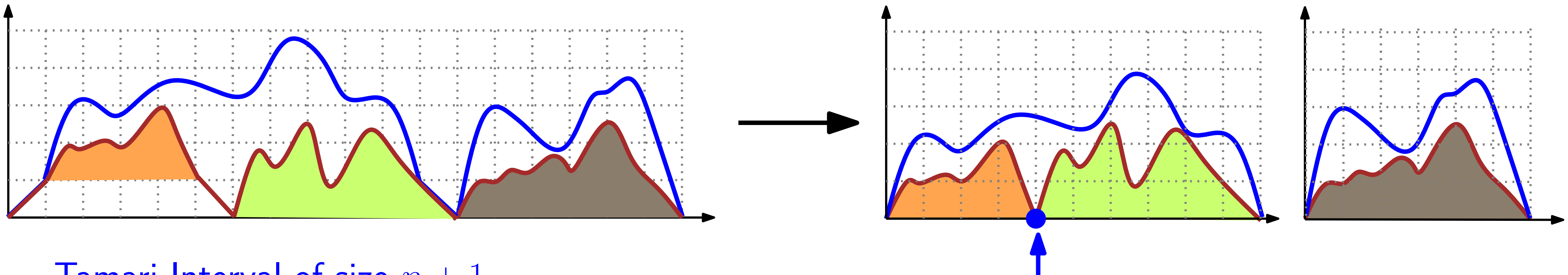

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$$F(x) = \frac{1+u}{(1+zu)(1-z)^3}(1-2z-z^2u) \quad , \qquad F(1) = \frac{1-2z}{(1-z)^3}$$

$$t = z(1-z)^3,$$
$$x = \frac{1+u}{(1+zu)^2}.$$

- Polynomial equation with one catalytic variable. Effective theory from Bousquet-Mélou–Jehanne, the solution is algebraic and explicit $\rightarrow$ Chapoton's theorem on enumeration.

Modest contribution: we add a marked point...

$H(x;t,s)$ series of interval with a marked point

$$H(x) \equiv H(t,x,s) := \sum_{n\geq 0} t^n \sum_{(P,Q)\in\mathcal{I}_n} x^{\mathrm{contact}(P)} \sum_{i=0}^{2n} s^{Q(i)}.$$

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$$H(x) = F(x) + sxt\frac{H(x)-H(1)}{x-1}F(x) + xt\frac{F(x)-F(1)}{x-1}H(x).$$

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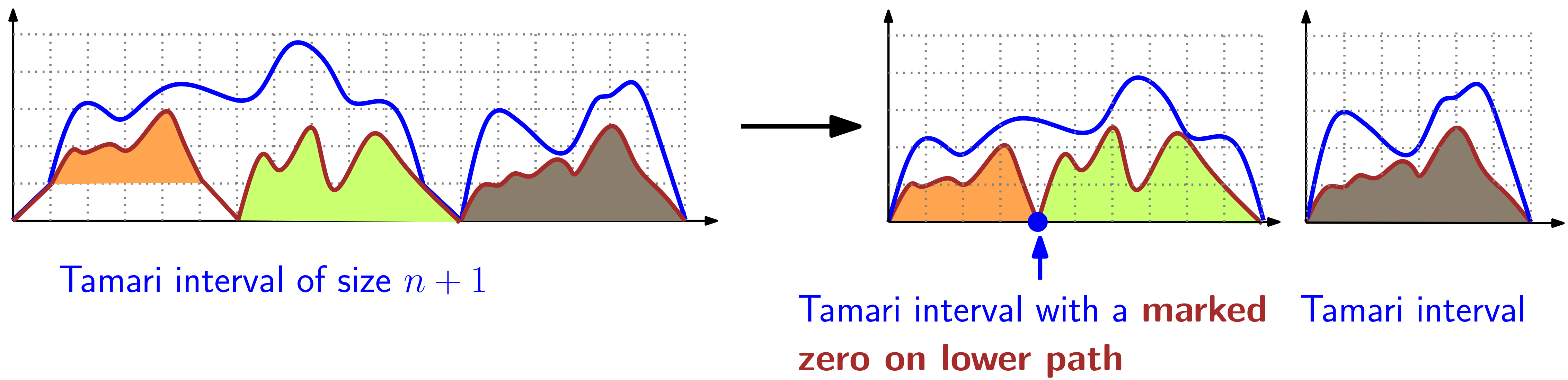

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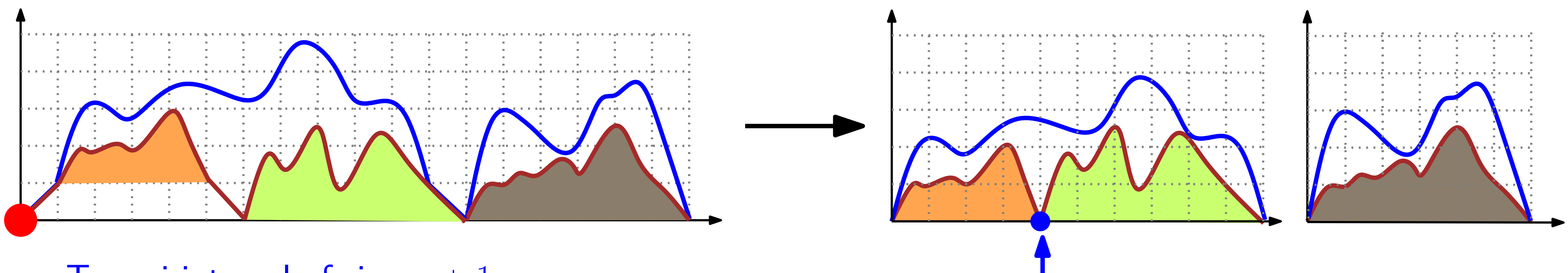

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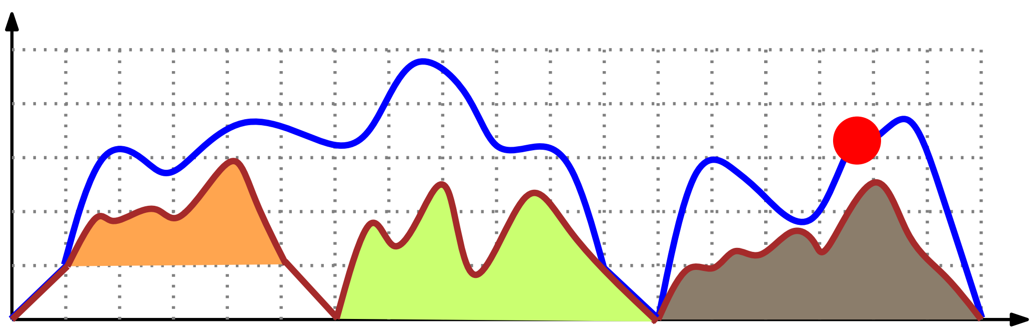

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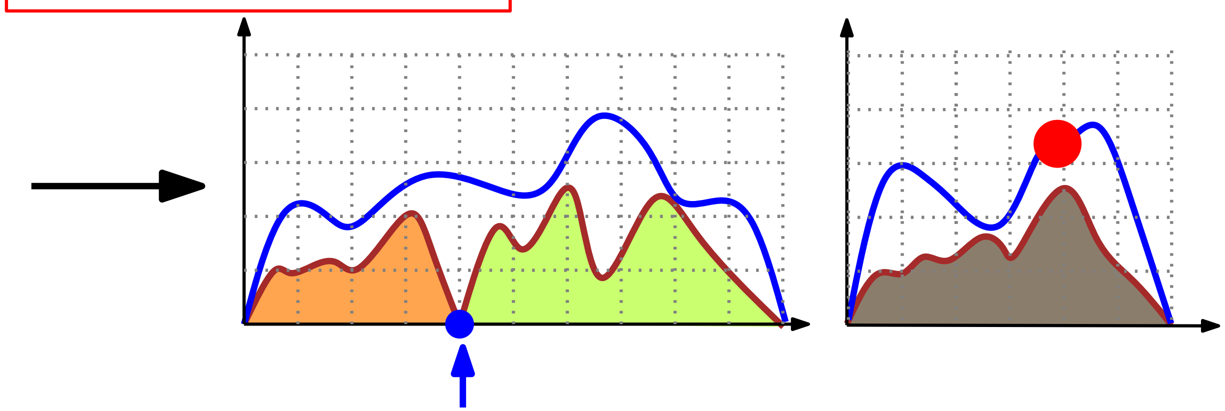

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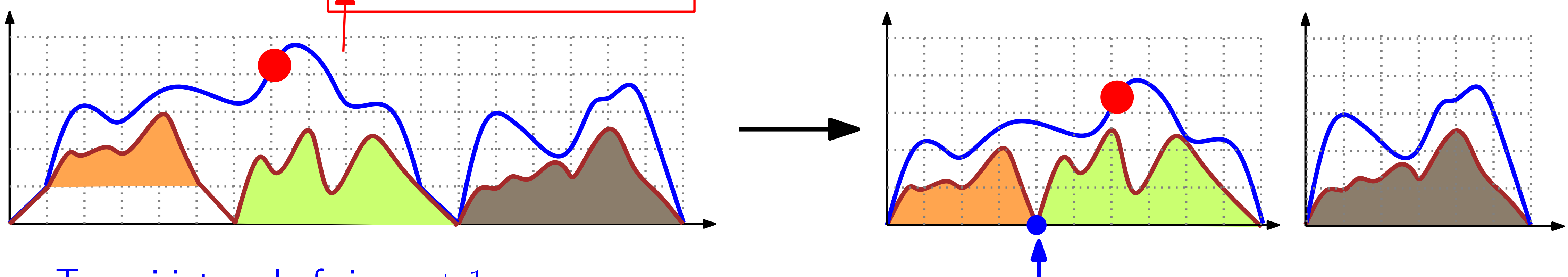

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If we know $F(x)$ (and we do) this is nothing but a (linear!) one-variable catalytic equation for H. This is solved "illico" with the kernel method!

Kernel method

$H(x;t,s)$ series of intervals with a marked point

$$H(x) = F(x) + {\color{red}s}xtF(x)\frac{H(x) - H(1)}{x-1} + xt\frac{F(x) - F(1)}{x-1}H(x).$$

Kernel method

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$$H(x) = F(x) + sxtF(x)\frac{H(x)-H(1)}{x-1} + xt\frac{F(x)-F(1)}{x-1}H(x).$$

We write: $$K(x)H(x) = F(x) - \frac{sxtF(x)H(1)}{(x-1)}$$

$$K(x) = \left(-1 + sxt\frac{F(x)}{x-1} + xt\frac{F(x)-F(1)}{x-1}\right).$$

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$H(x;t,s)$ series of intervals with a marked point

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$\longrightarrow$ This result contains in principle the whole distribution of the random variable $Q_n(I)$.... But how to deduce the wanted asymptotic?!?

$$h(t,s) := \sum_{n\geq 0} t^n \sum_{(P,Q)\in\mathcal{I}_n} \sum_{i=0}^{2n} s^{Q(i)}.$$

Transfer theorem and D-finiteness...

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• **The trick – the real good thing in this work :)**

Every algebraic series is D-finite (solution of a linear DE with polynomial coeffs)

Our series, seen in the variable s, and even $(s-1)$, is algebraic over algebraic over $\mathbb{Q}(t)$

Therefore it is D-finite: its coefficients, the h_k, satisfy a polynomial recurrence relation!

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where $c_k = \frac{\sqrt{6}(3k-4)(3k-8)}{96} c_{k-2}$. The recurrence relation is immediately solved and leads to the theorem in the first slide!

$$\mathbf{E}[Z^k] = \frac{\sqrt{3}\cdot 2^{-\frac{k}{4}-1}}{\sqrt{\pi}} \frac{\Gamma(\frac{1}{4}k+\frac{1}{3})\Gamma(\frac{1}{4}k+\frac{2}{3})}{\Gamma(\frac{1}{4}k+\frac{1}{2})}.$$

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if not already known, this trick is the most interesting thing in my paper

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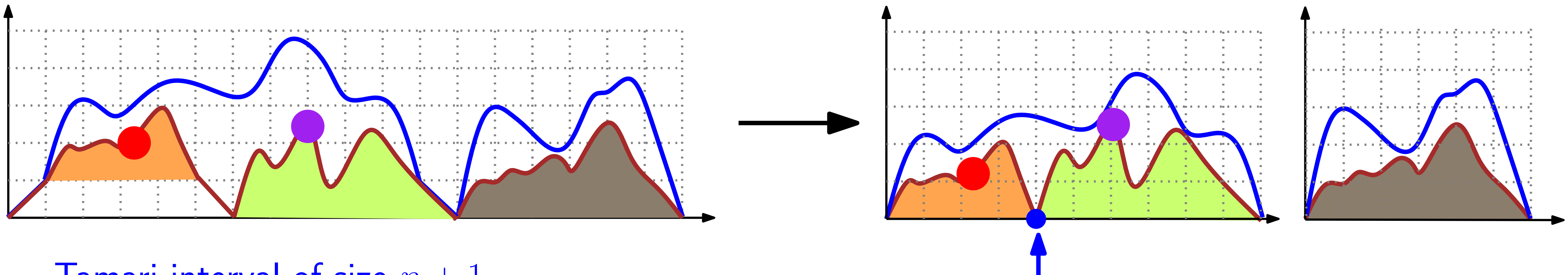

Tamari interval of size $n+1$

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Tamari interval

● ● $\rightarrow$ for the lower path, we need to know if the maked point comes before or after the first contact, otherwise we can't follow its height!

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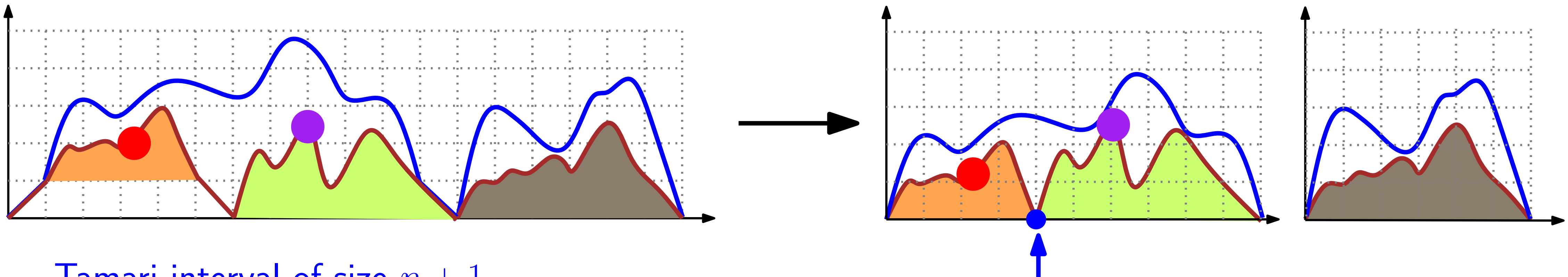

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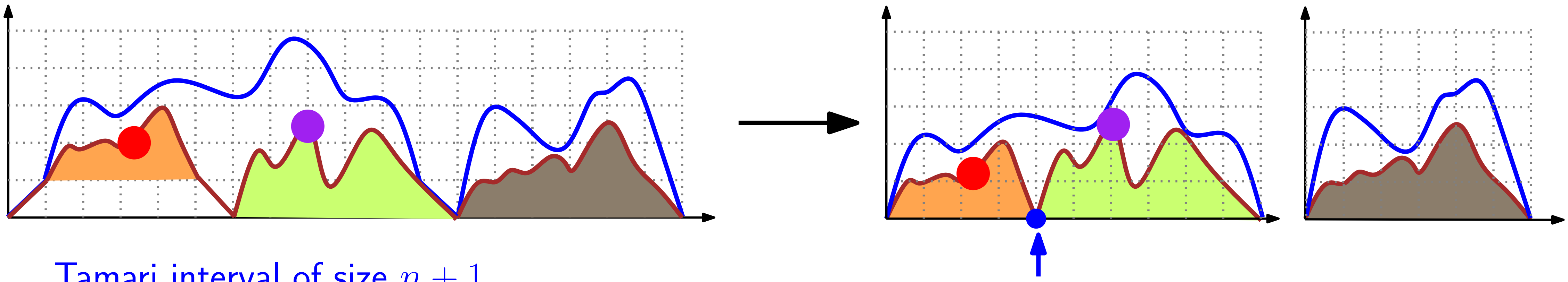

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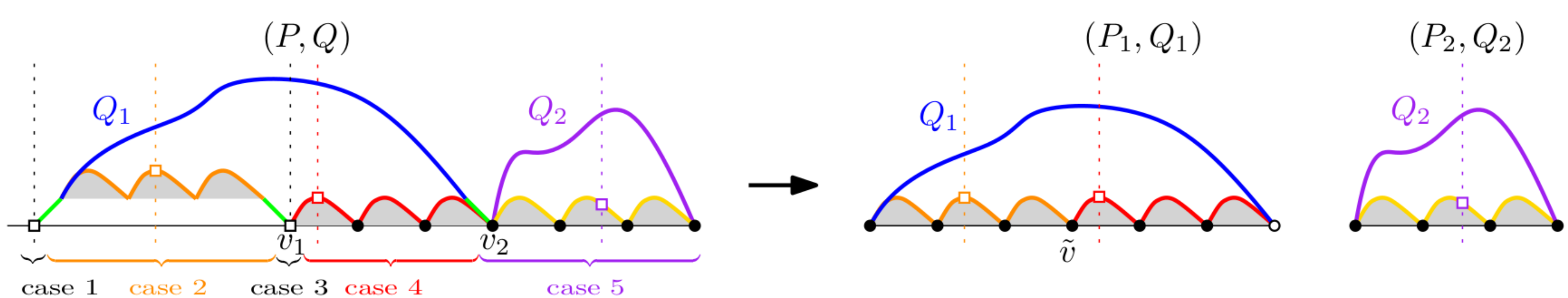
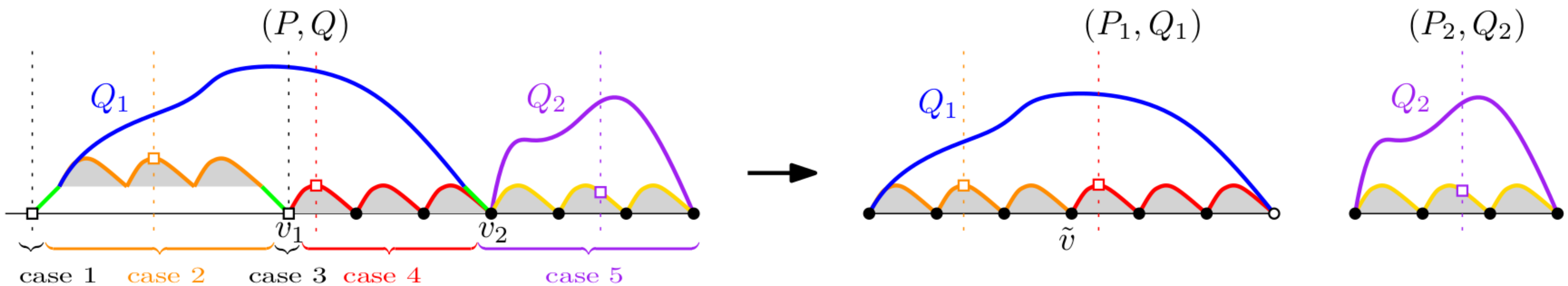


Two catalytic variables... but not for real



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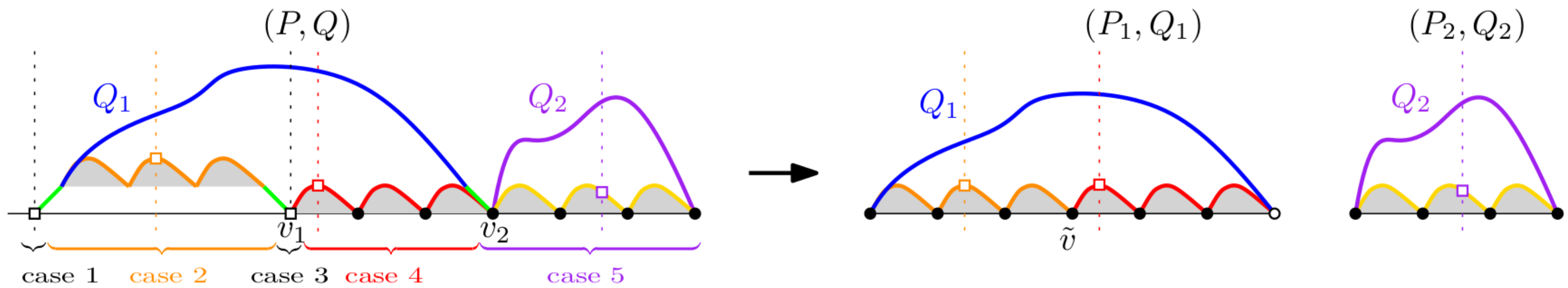

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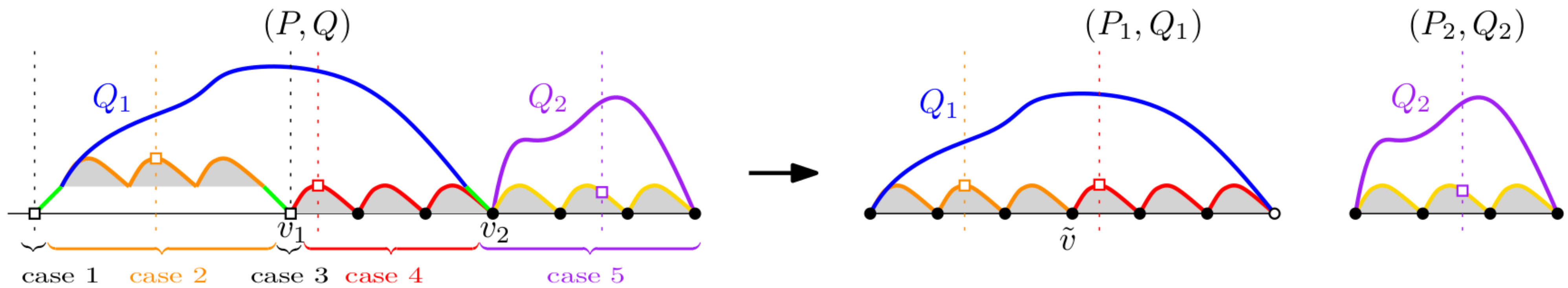

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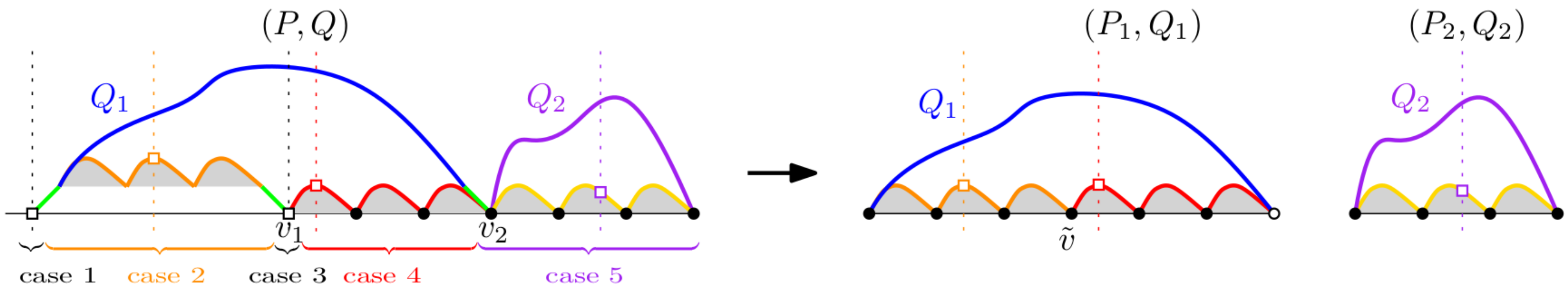

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$\rightarrow$ For the asymptotic, the "D-finite trick" again works!

Conclusion

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- I'd like to have more applications of my D-finite trick!

Thanks!