

Discrete Whittaker processes

Neil O'Connell

University College Dublin

Cartipon 25, Le Croisic

June 11, 2025

Reverse plane partitions

$$\pi = (\pi_{ij}, (i,j) \in \lambda) \in \mathbb{Z}_+^{\lambda}$$

$$\pi_{ij} \geq \pi_{i,j-1} \vee \pi_{i-1,j}$$

$$\text{Convention: } \pi_{i0} = \pi_{0j} = 0$$

Reverse plane partition
of shape 3211

0	1	1
2	4	
2		
3		

$$\text{RPP}(\lambda) = \text{RPP's of shape } \lambda$$

Markov chain on RPP(1)

Subtract one from π_{ij} at rate

$$b_{ij}(\pi) = (\pi_{ij} - \pi_{i-1,j})(\pi_{ij} - \pi_{i,j-1})$$

$$\text{conv. } \pi_{10} = \pi_{0j} = 0$$

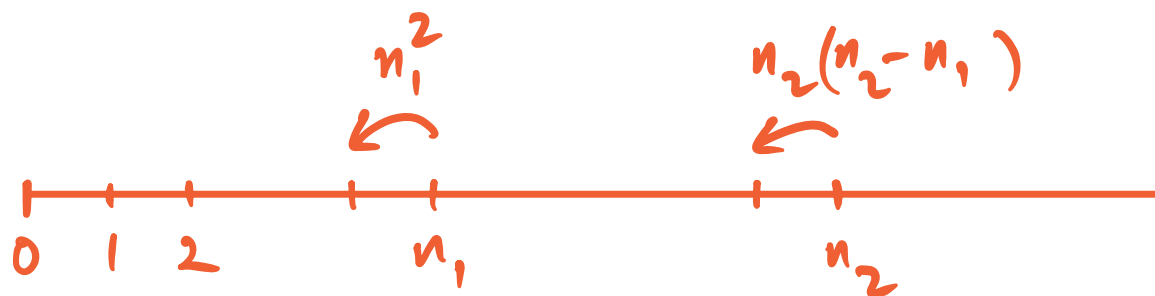
Generator:
$$G^1 = \sum_{(i,j) \in \pi} b_{ij}(\pi) D_{\pi_{ij}}$$

$$D_n f(n) = f(n-1) - f(n)$$

Some properties

1. Nested: if $\pi \sim G^\lambda$ and $\mu \subset \lambda$,
then $\pi|_\mu \sim G^\mu$.

In particular, first row $n_j = \pi_{1j}$:

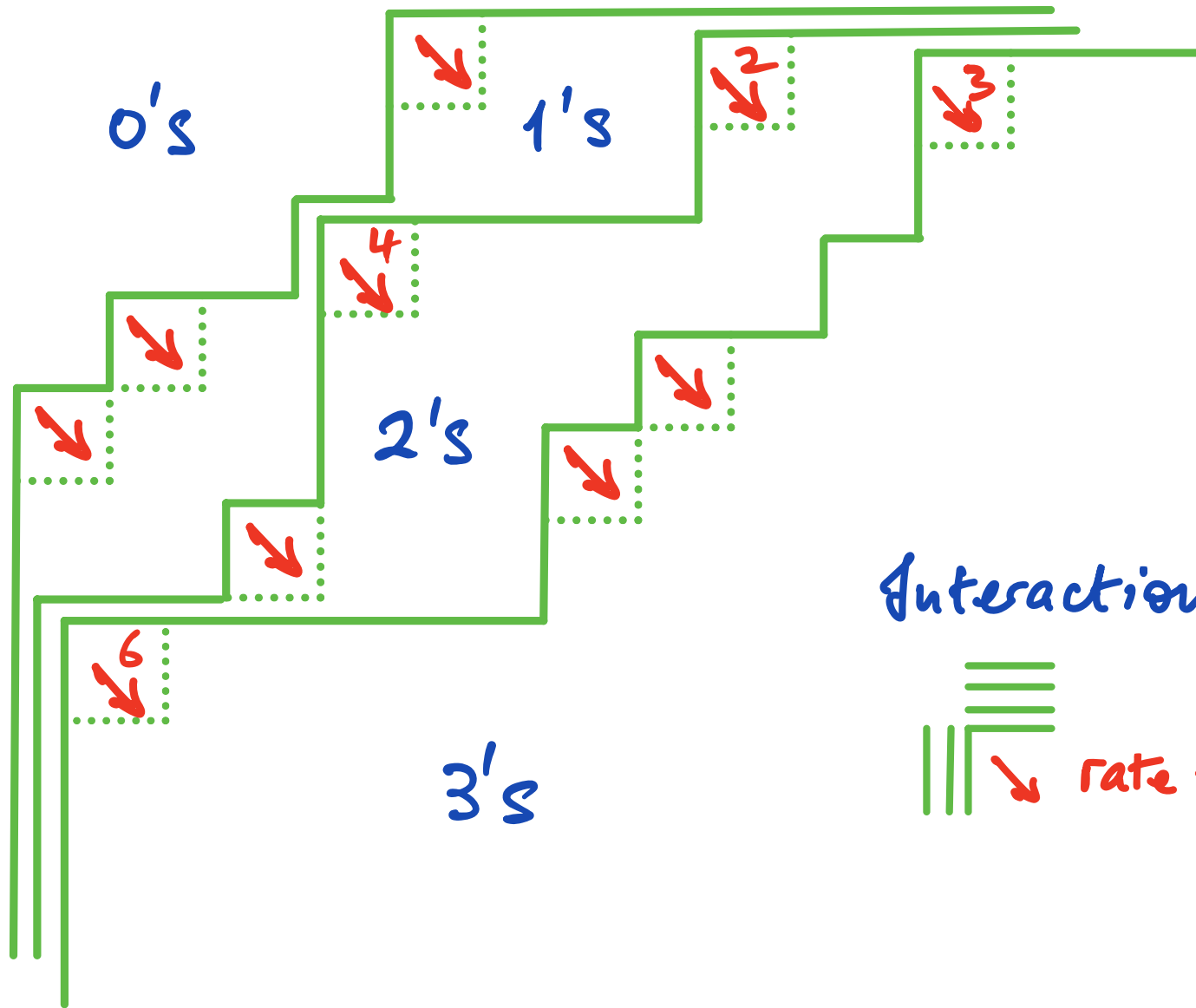


n_1	n_2	n_3

2. Theorem: Unique entrance law from
 $\pi_{ij} = +\infty \quad \forall i, j \in \lambda$

3. Can take $\lambda = \mathbb{N}^2$

As a system of interacting corner growth processes



$$\text{Shape 1} = N^2$$

Interaction :

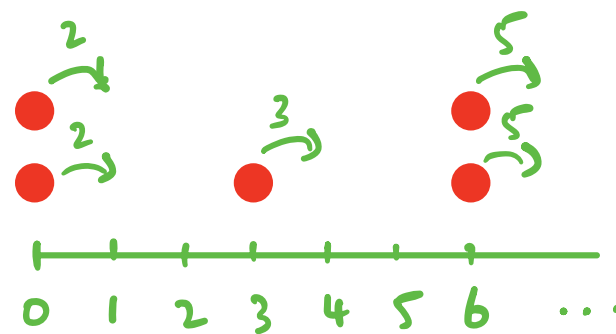
The diagram shows a corner where three green lines meet. A red arrow points from the corner into the region. The text next to it is:

$$\text{rate} = 3 \times 4 = 12$$

Discrete (repulsive) δ -Bose gas

First row (or column) :

Independent Poisson processes
with rank dependent rates



Fwd diff. op.

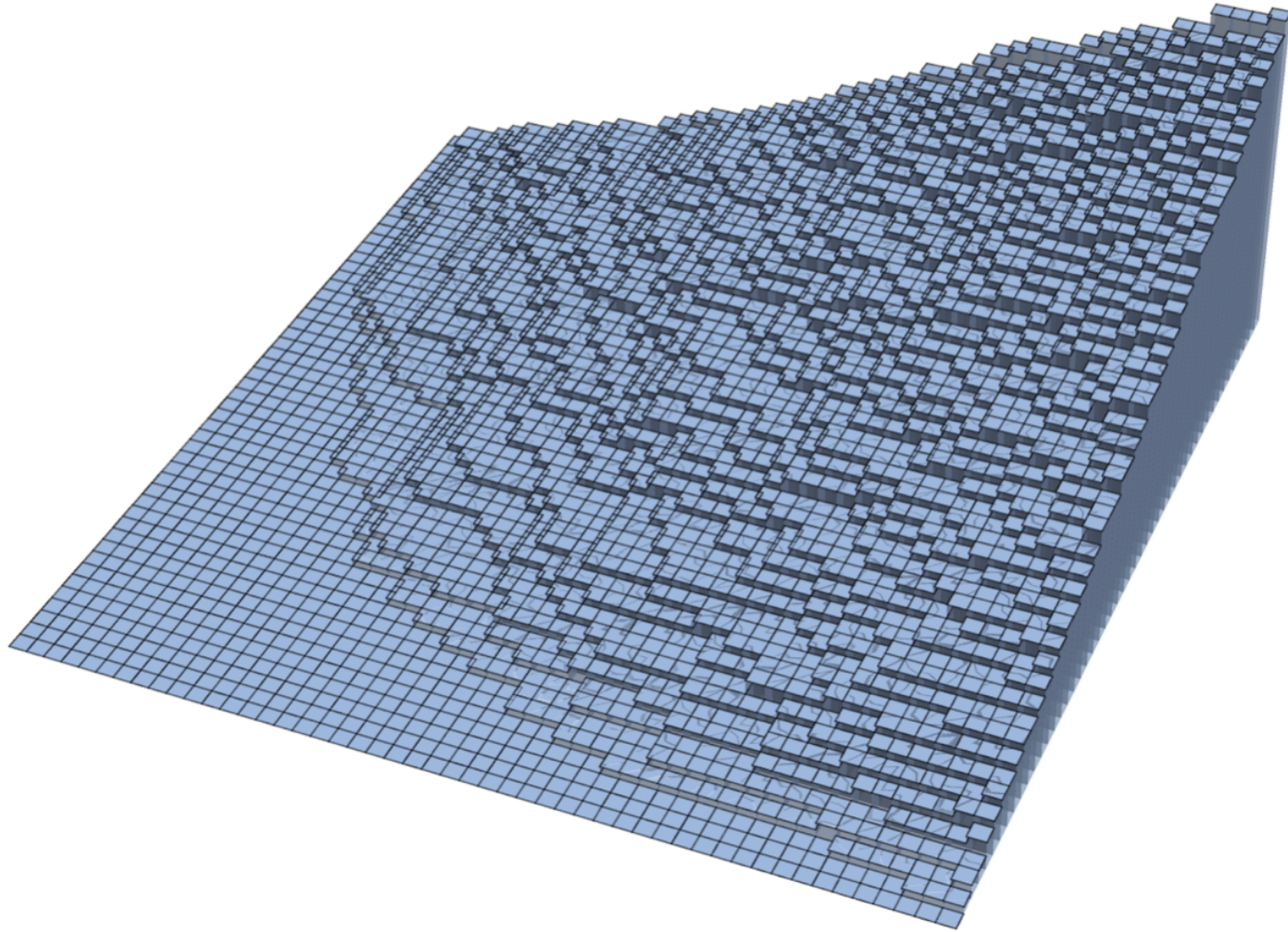


Doob transform of $\sum B_{x_i} - \sum_{i \neq j} \mathbb{1}_{x_i = x_j}$

via $\zeta(x) = \prod_i i^{x(i)}$

$$x_{(1)} \leq x_{(2)} \leq \dots$$

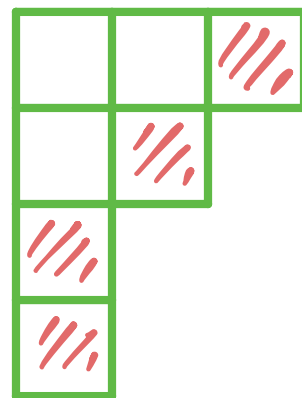
Simulation



Restrictions to skew-shapes

Let $\mu \subset \lambda^0 = \{(i, j) \in \lambda : (i+1, j), (i, j+1) \in \lambda\}$

eg. $\lambda = 3211$,
 $\mu = \lambda^0 = 21$



λ/μ

Then if initial law chosen correctly
and $\pi \sim \mathcal{G}^\lambda$ then $\pi|_{\lambda/\mu}$ is Markov.

Example

$$I = 21, \mu = 1$$

π_{11}	π_{12}
π_{21}	

k	n
m	

$$Q = k^2 D_k + n(n-k) D_n + m(m-k) D_m$$

Suppose that, initially, $\pi_{12} = n$, $\pi_{21} = m$

and $\pi_{11} \sim \binom{n+m}{n}^{-1} \binom{n}{k} \binom{m}{k}$.

Then, if $\pi \sim Q$, $\pi|_{I/\mu} \equiv (\pi_{12}, \pi_{21})$ is

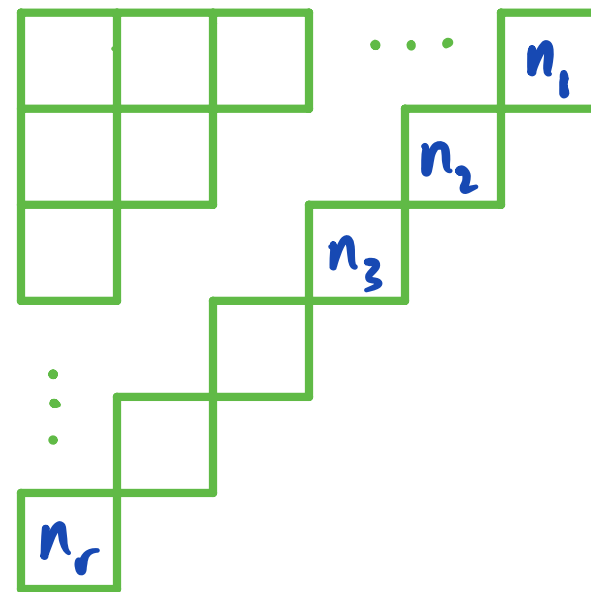
Markov w/ gen.

$$L = \frac{n^3}{n+m} D_n + \frac{m^3}{n+m} D_m.$$

Staircase shapes

$$\lambda = \delta_{r+1} = (r, r-1, \dots, 1)$$

$$\mu = \delta_r \quad \pi|_{\lambda/\mu} \in \mathbb{Z}_+^r.$$



$$A_r(n) = \sum_{\substack{\pi \in \text{RPP}(\lambda) \\ \pi|_{\lambda/\mu} = n}} \prod_{(i,j) \in \delta_r} \left(\frac{\pi_{i+1,j}}{\pi_{i,j}} \right) \left(\frac{\pi_{i,j+1}}{\pi_{i,j}} \right)$$

$W_r(\pi)$

Theorem Suppose that, initially, either

$$\pi|_{\mathcal{A}/\mu} = n \quad \text{and} \quad \pi \sim A_r(n) \bar{\gamma}' W_r(\pi)$$

$$\text{or} \quad \pi_{ij} = +\infty \quad \forall i, j \in \mathcal{A} \quad .$$

Then, if $\pi \sim G^{\mathcal{A}}$, $\pi|_{\mathcal{A}/\mu}$ is a Markov chain on $\mathbb{Z}_+^r$ with generator

$$L^r = \sum_{i=1}^r \frac{A_r(n - e_i)}{A_r(n)} n_i^2 D_{n_i}$$

$$\equiv A_r(n) \bar{\gamma}' \circ H^r \circ A_r(n), \quad \text{where} \dots$$

Fundamental Whittaker Functions

$$\phi_r(x) = \sum_{n \in \mathbb{Z}_+^r} \frac{A_r(n)}{\prod_{i=1}^r n_i!^2} \prod_{i=1}^r e^{n_i(x_i - x_{i+1})}$$

satisfies $\mathcal{H}^r \phi_r = 0$ where

$$\mathcal{H}^r = -\frac{1}{2} \sum_{i=1}^{r+1} \frac{\partial^2}{\partial x_i^2} + \sum_{i=1}^r e^{x_i - x_{i+1}}$$

Equivalently, $H^r A_r = 0$ where

$$H^r = \sum_{i=1}^r n_i^2 D_{n_i} + \sum_{i=1}^{r-1} n_i n_{i+1}.$$

Hashizume (1982), Ishii-Stade (2007)

Givental (1997), Batyrev et al (2000)

Absorption times $r=1, 2$

$$r=1, \quad L = n^2 D_n$$

Absorption time from n :
$$\tau_n = \sum_{k=1}^n k^2$$

$$r=2, \quad L = \frac{n^3}{n+m} D_n + \frac{m^3}{n+m} D_m$$

Absorption time from (n, m) :

$$\tau_{n,m} \stackrel{d}{=} \tau_n + \tau'_m$$

↑ indep. copy
of τ_m

Hitting probabilities ($r=2$)

$$S_p = - \sum_{n=1}^{\infty} \frac{n^p q^n}{(1-q^n)^2}$$

primitive
cube root of
unity

$$q = e^{2\pi i \omega} = -e^{-\pi\sqrt{3}}$$

Ramanujan: $S_2 = \frac{1}{24\pi^2}$, $S_4 = \frac{3 \Gamma(1/3)^{18}}{40960 \pi^{12}}$, ...

$$P_{\infty}(T_{ij} < \infty) = \text{lin. comb. of } S_p \text{'s.}$$

$$P_{\infty}(T_{01} < \infty) = 12\pi^2 S_2 = \frac{1}{2}$$

$$P_{\infty}(T_{11} < \infty) = 24\pi^2 S_5 \approx 0.878..$$

Imaginary exponential fctls

$B_1, B_2, \dots, B_{r+1}$ indep. BM's

$$Y_k(t) = e^{i(B_k(t) - B_k(0))}, \quad Z_k(t) = \int_0^t Y_k(s) ds \quad k=1, \dots, r.$$

Transition probabilities for L^r process:

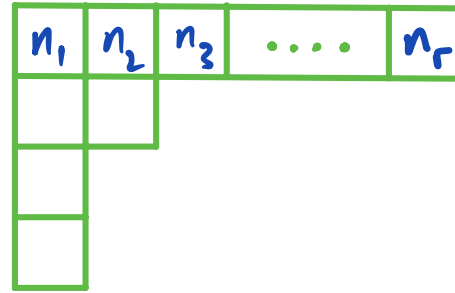
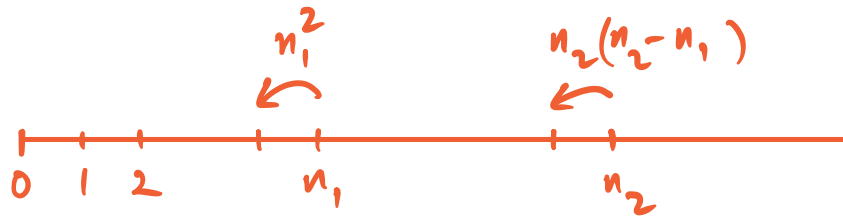
$$p_t(n, m) = \frac{n!^2}{m!^2 (n-m)!} \frac{A_r(m)}{A_r(n)} \mathbb{E} Y(t)^{-n} Z(t)^{n-m}$$

$$\mathbb{P}_n(T_0 \leq t) = p_t(n, 0) = \frac{n!}{A_r(n)} \mathbb{E} Z(t)^n.$$

$$n! = n_1! \dots n_r! \quad y^n = y_1^{n_1} \dots y_r^{n_r} \text{ etc.}$$

Semidiscrete polymer with imag. disorder

Evolution of first row :



Transition kernel $q_t(n, m)$ $n, m \in \mathbb{Z}_+^r$

$$q_t(\underbrace{(n, n, \dots, n)}_r, (0, 0, \dots, 0)) = \frac{1}{n!^r} E \mathcal{U}_r(t)^n$$

$$\mathcal{U}_r(t) = \int_{0 < s_1 < s_2 < \dots < s_r < t} \gamma_1(s_1) \dots \gamma_r(s_r) ds_1 \dots ds_r$$