Lusztig's asymptotic algebra in type $\tilde{A}_n$

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§1. Kazhdan-Lusztig Theory basics

- (W,S) a Coxeter system, $\ell : W \to \mathbb{N}$ length function.
- $R = \mathbb{Z}[q, q^{-1}]$
- The Hecke algebra is the algebra H over R with basis $\{T_w \mid w \in W\}$ and relations

$$T_w T_s = \begin{cases} T_{ws} & \text{if } \ell(ws) > \ell(w) \\ T_{ws} + (q - q^{-1}) T_w & \text{if } \ell(ws) < \ell(w) \end{cases}$$

- "Bar involution" $\bar{q} = q^{-1}$ on R extends to H

$$\overline{\sum a_w T_w} = \sum \overline{a_w}\, T_{w^{-1}}^{-1}$$

Theorem (Kazhdan, Lusztig, 79) There exists a unique basis $(C_w)_{w \in W}$ of H such that

- $\overline{C_w} = C_w$
- $C_w = T_w + \sum_{v < w} P_{v,w}(q^{-1}) T_v$

where $P_{v,w}(q^{-1}) \in q^{-1}\mathbb{Z}[q^{-1}]$.

$(C_w)_{w \in W}$ is the KL-basis

$P_{v,w}(q^{-1})$ are the KL-polynomials.

Type A_2 Simplified mult. table $C_x C_y = \sum h_{xyz} C_z$

$x \backslash y$	e	s	ts	t	st	sts
e	e	s	ts	t	st	sts
s	s	s	s, sts	st	st	sts
st	st	s, sts	s, sts	st	st, sts	sts
t	t	ts	ts	t	t, sts	sts
ts	ts	ts	ts, sts	t, sts	t, sts	sts
sts	sts	sts	sts	sts	sts	sts

Left Kazhdan-Lusztig cells

$x \backslash y$	e	s	ts	t	st	sts
e	e	s	ts	t	st	sts
s	s	s	s, sts	st	st	sts
st	st	s, sts	s, sts	st	st, sts	sts
t	t	ts	ts	t	t, sts	sts
ts	ts	ts	ts, sts	t, sts	t, sts	sts
sts	sts	sts	sts	sts	sts	sts

Γ a left cell $\rightsquigarrow$ $|\Gamma|$-dim rep of H.

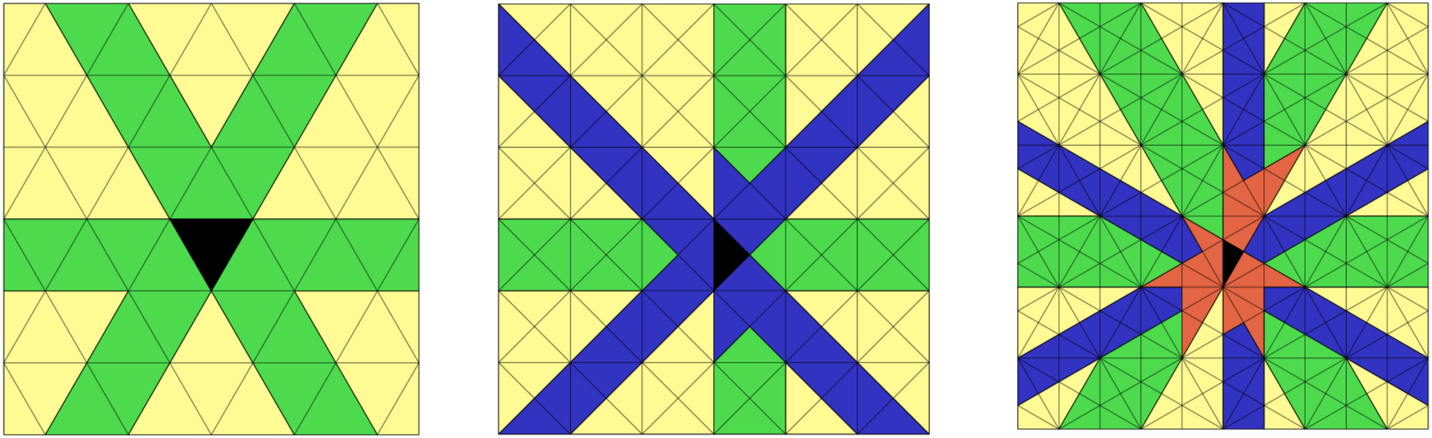
Left cells

Right cells

x \ y	e	s	ts	t	st	sts
e	e	s	ts	t	st	sts
s	s	s	s, sts	st	st	sts
st	st	s, sts	s, sts	st	st, sts	sts
t	t	ts	ts	t	t, sts	sts
ts	ts	ts	ts, sts	t, sts	t, sts	sts
sts	sts	sts	sts	sts	sts	sts
	e	s, t, st, ts				sts

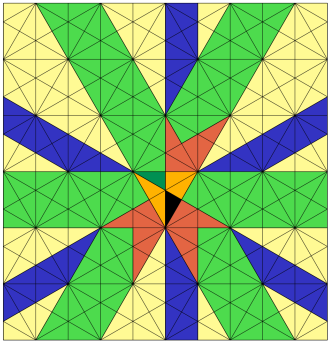
Two sided cells

Cell decomposition affine rank 2 (Lusztig)

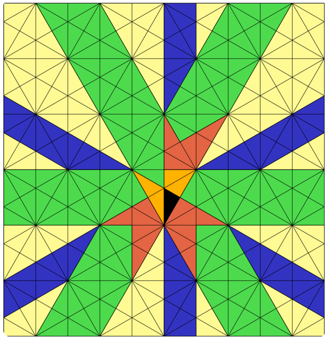


Remark: There is a generalisation of KL-theory to "unequal parameters." In this case the theory is less developed — the geometric interpretations of "equal parameter" case not available...

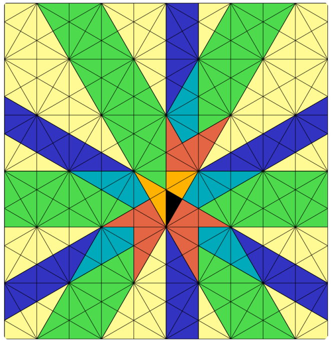
__Example__ $\tilde{G}_2$ with parameters $\overset{b}{\vert}\!\!-\!\!\!-\!\!\overset{b}{\vert}\!\!\overset{6}{-\!\!\!-}\!\!\overset{a}{\vert}$ (J. Guilhot)



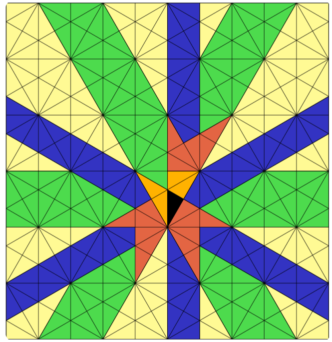
$\frac{a}{b} > 2$



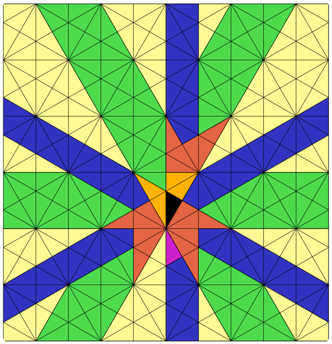
$\frac{a}{b} = 2$



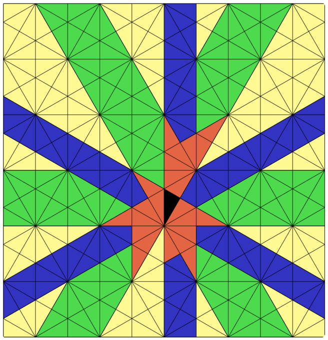
$2 > \frac{a}{b} > \frac{3}{2}$



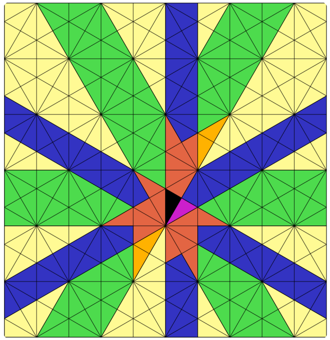
$\frac{a}{b} = \frac{3}{2}$



$\frac{3}{2} > \frac{a}{b} > 1$



$\frac{a}{b} = 1$



$\frac{a}{b} < 1$

We now return to the A_2 example..

Lusztig's $\underset{\sim}{a}$-function: $\boxed{\underset{\sim}{a}(z) = \max\{\deg h_{xyz} \mid x, y \in W\}}$

$x \backslash y$	e	s	ts	t	st	sts
e	$\overset{0}{e}$	$\overset{0}{s}$	$\overset{0}{ts}$	$\overset{0}{t}$	$\overset{0}{st}$	$\overset{0}{sts}$
s	$\overset{0}{s}$	$\overset{1}{s}$	$\overset{0}{s}, \overset{0}{sts}$	$\overset{0}{st}$	$\overset{1}{st}$	$\overset{1}{sts}$
st	$\overset{0}{st}$	$\overset{0}{s}, \overset{0}{sts}$	$\overset{1}{s}, \overset{1}{sts}$	$\overset{1}{st}$	$\overset{0}{st}, \overset{1}{sts}$	$\overset{2}{sts}$
t	$\overset{0}{t}$	$\overset{0}{ts}$	$\overset{1}{ts}$	$\overset{1}{t}$	$\overset{0}{t}, \overset{0}{sts}$	$\overset{1}{sts}$
ts	$\overset{0}{ts}$	$\overset{1}{ts}$	$\overset{0}{ts}, \overset{1}{sts}$	$\overset{0}{t}, \overset{0}{sts}$	$\overset{1}{t}, \overset{1}{sts}$	$\overset{2}{sts}$
sts	$\overset{0}{sts}$	$\overset{1}{sts}$	$\overset{2}{sts}$	$\overset{1}{sts}$	$\overset{2}{sts}$	$\overset{3}{sts}$

$$\underset{\sim}{a}(st) = \max\{0, 1\} = 1$$

$x \backslash y$	e	s	ts	t	st	sts
e	$\overset{0}{e}$	$\overset{0}{s}$	$\overset{0}{ts}$	$\overset{0}{t}$	$\overset{0}{st}$	$\overset{0}{sts}$
s	$\overset{0}{s}$	$\overset{1}{s}$	$\overset{0}{s}, \overset{0}{sts}$	$\overset{0}{st}$	$\overset{1}{st}$	$\overset{1}{sts}$
st	$\overset{0}{st}$	$\overset{0}{s}, \overset{0}{sts}$	$\overset{1}{s}, \overset{1}{sts}$	$\overset{1}{st}$	$\overset{0}{st}, \overset{1}{sts}$	$\overset{2}{sts}$
t	$\overset{0}{t}$	$\overset{0}{ts}$	$\overset{1}{ts}$	$\overset{1}{t}$	$\overset{0}{t}, \overset{0}{sts}$	$\overset{1}{sts}$
ts	$\overset{0}{ts}$	$\overset{1}{ts}$	$\overset{0}{ts}, \overset{1}{sts}$	$\overset{0}{t}, \overset{0}{sts}$	$\overset{1}{t}, \overset{1}{sts}$	$\overset{2}{sts}$
sts	$\overset{0}{sts}$	$\overset{1}{sts}$	$\overset{2}{sts}$	$\overset{1}{sts}$	$\overset{2}{sts}$	$\overset{3}{sts}$

$\underset{\sim}{a}(e) = 0$ $\underset{\sim}{a}(s) = \underset{\sim}{a}(t) = \underset{\sim}{a}(st) = \underset{\sim}{a}(ts) = 1$ $\underset{\sim}{a}(sts) = 3$

Fact: $\underset{\sim}{a}$-function is constant on two-sided cells

Define $\gamma_{x,y,z^{-1}} \in \mathbb{Z}$ (possibly zero) by:

$$h_{x,y,z} = q^{a(z)} \gamma_{x,y,z^{-1}} + \text{lower degree}$$

Lusztig's asymptotic algebra is $\mathbb{Z}$-algebra

$$\boxed{H^\infty = \mathbb{Z}\text{-span}\{\tau_w \mid w \in W\}}$$

with multiplication

$$\tau_x \tau_y = \sum \gamma_{x,y,z^{-1}} \tau_z$$

It is a deep fact that H^∞ is an associative algebra, making use of geometric interpretations of KL-theory

$x \setminus y$	τ_e	τ_s	τ_{ts}	τ_t	τ_{st}	τ_{sts}
τ_e	τ_e	0	0	0	0	0
τ_s	0	τ_s	0	0	τ_{st}	0
τ_{st}	0	0	τ_s	τ_{st}	0	0
τ_t	0	0	τ_{ts}	τ_t	0	0
τ_{ts}	0	τ_{ts}	0	0	τ_t	0
τ_{sts}	0	0	0	0	0	τ_{sts}

$$H^{\infty} \simeq \mathbb{Z} \oplus \mathrm{Mat}_{2,2}(\mathbb{Z}) \oplus \mathbb{Z}$$

- We have $H^\infty = \bigoplus_{\lambda \in \Lambda} H^\infty_\lambda$ (Λ indexes 2-sided cells)

- Constructing H^∞ extremely difficult! Requires:
  - detailed understanding of cell decomp.
  - calculation of $\underline{a}$-function
  - understanding γ_{xyz} coefficients

  very few cases where this level of detail is known

- In general parameters, only conjectural (Lusztig) that H^∞ is an associative algebra.

  $\rightsquigarrow$ New techniques are required

The asymptotic algebra in type $\tilde{A}_n$

- Shi and Lusztig:

$$\begin{array}{ccccc} & & \Lambda & & \\ & & \shortparallel & & \\ \{\text{two-sided cells}\} & \overset{\text{bij}}{\longleftrightarrow} & \{\text{partitions of } n+1\} & \overset{\text{bij}}{\longleftrightarrow} & \left\{\begin{array}{l}\text{unipotent conj.}\\ \text{classes } GL_{n+1}(\mathbb{C})\end{array}\right\} \\ \Delta_\lambda & \longleftarrow & \lambda & \longrightarrow & u_\lambda \ (\text{Jordan}) \end{array}$$

Theorem (Xi 2002) For $\lambda = (\lambda_1, \lambda_2, \dots) \in \Lambda$,

$$H_\lambda^\infty \simeq \mathrm{Mat}_{N_\lambda, N_\lambda}(\mathrm{Rep}(F_\lambda)) \qquad \left(N_\lambda = \frac{(n+1)!}{\lambda_1! \lambda_2! \dots} \right)$$

where F_λ = maximal reductive subgroup of the centraliser $C(u_\lambda)$ in $GL_{n+1}(\mathbb{C})$.

- Xi's proof uses detailed description of cells (Shi) along with intricate calculations in H.
- Another approach (Kim, Pylyavskyy 2023) to H^∞ has been given via "affine matrix ball construction" (affine RSK)
- These approaches are both specific to "type A".
- The purpose of this talk is to outline a new approach to constructing H^∞ in $\tilde{A}_n$ that is adaptable to other types (and also to unequal parameters).

In our approach:

- $\mathrm{Rep}(F_\lambda)$ arises combinatorially via Schur functions
- The fundamental λ-alcove is central to the technique — and H^∞ is constructed combinatorially using λ-folded alcove paths (an analogue of Ram's alcove paths and Littelmann Path Model).
- Required information on cells / KL-theory is minimised (we obtain new description of cells in type $\tilde{A}_n$).

§2. Summary of strategy

- We construct a family $(\pi_\lambda)_{\lambda \in \Lambda}$ of matrix representations which form a balanced system of cell representations.

  $\rightsquigarrow$ This implies that

  "leading matrices"

  $$H_\lambda^\infty \simeq \langle \text{top degree slice of matrices } \pi_\lambda(T_w) \rangle$$

- We develop a combinatorial formula for the matrix entries of $\pi_\lambda(T_w)$ $\rightsquigarrow$ combined with other ingredients gives explicit description of leading matrices, hence H_λ^∞.

Family $(\pi_\lambda)_{\lambda \in \Lambda}$ of reps of H

$\lambda \in \Lambda \rightsquigarrow J_\lambda \subseteq \{1, \dots, n\}$ as follows:

$$\lambda = \begin{array}{|c|c|c|c|c|c|c|} \hline 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 8 & 9 & 10 & 11 & 12 \\ \cline{1-5} 13 & 14 & 15 \\ \cline{1-3} 16 & 17 & 18 \\ \cline{1-3} 19 & 20 \\ \cline{1-2} \end{array} \rightsquigarrow J_\lambda \text{ in green.}$$

For $\lambda \in \Lambda$ let $\pi_\lambda = \operatorname{Ind}_{L_\lambda}^{H}(\psi_\lambda)$ where

$$\psi_\lambda = \left[\begin{array}{l} \text{"generic" 1-dim rep of } J_\lambda\text{-Levi subalgebra } L_\lambda \\ \text{defined over a polynomial ring } R[z_\lambda] \end{array} \right]$$

We fix a natural basis, making π_λ a matrix representation

$$\pi_\lambda(T_w) \in \operatorname{Mat}_{N_\lambda, N_\lambda}(R[z_\lambda])$$

Killing property

Theorem (Killing property)

$\pi_\lambda(C_w) = 0$ whenever w is in a lower or incomparable cell to Δ_λ

- Essentially says that π_λ realises the quotienting process, killing lower cells.

Boundedness Property

Write $\deg \pi_\lambda(T_w)$ for maximal degree (in q) of matrix entries of $\pi_\lambda(T_w)$

Theorem (Boundedness) For $\lambda \in \Lambda$ we have

$$\deg \pi_\lambda(T_w) \leq \ell(w_{\lambda'}) \qquad \forall w \in W$$

Consequence We define leading matrices

$$c_\lambda(w) = q^{-\ell(w_{\lambda'})} \pi_\lambda(T_w) \Big|_{q^{-1}=0}$$

Hence $\quad c_\lambda(w) \in \mathrm{Mat}_{N_\lambda, N_\lambda}(\mathbb{Z}[z_\lambda])$

Recognising cells

Our representations recognise two-sided cells:

Theorem (Recognising cells) For $\lambda \in \Lambda$ we have

$$\Delta_\lambda = \{ w \in W \mid \deg \pi_\lambda(T_w) = \ell(w_{\lambda'}) \}$$

The direction

$$\deg \pi_\lambda(T_w) = \ell(w_{\lambda'}) \implies w \in \Delta_\lambda$$

is relatively straight forward

The reverse implication is more subtle...

(... asymptotic Plancherel formula)

Realising the asymptotic algebra

A consequence of the above is:

Theorem (Representation of H_λ^∞) For $\lambda \in \Lambda$ we have

$$H_\lambda^\infty \simeq \langle c_\lambda(w) \mid w \in \Delta_\lambda \rangle_{\mathbb{Z}}, \qquad \tau_w \mapsto c_\lambda(w)$$

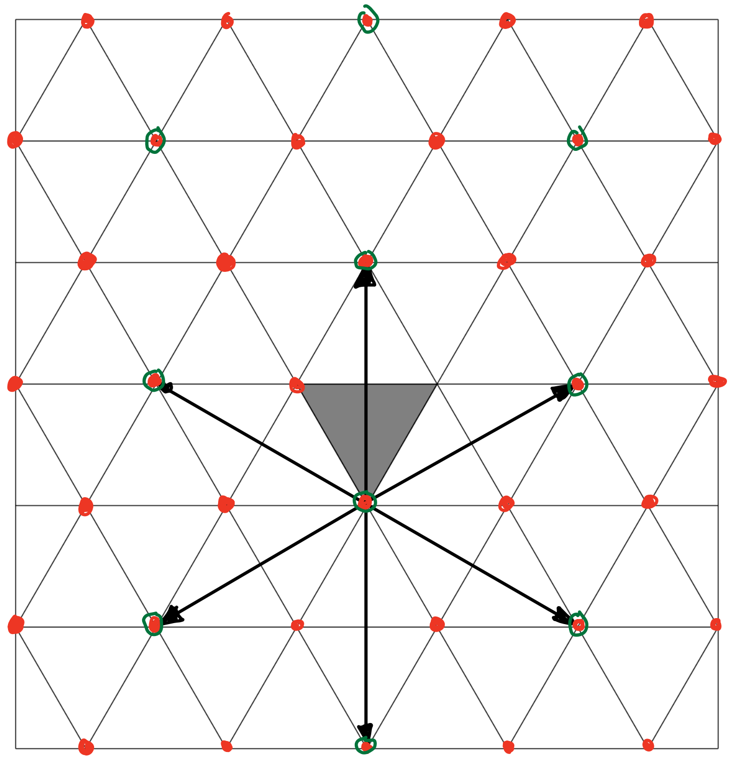
Thus H_λ^∞ is realised as a subalgebra of

$$\mathrm{Mat}_{N_\lambda, N_\lambda}(\mathbb{Z}[z_\lambda])$$

We must now explicitly compute the leading matrices...

Determining the leading matrices

- Φ root system type A_n
- Q coroot lattice
- P coweight lattice
- $W_0 \cong S_{n+1}$ Weyl gp
- $W = P \rtimes W_0$ extended affine Weyl group
- $\mathcal{A}_0$ = fundamental alcove



Hyperplanes $H_{\alpha,k}$ ($\alpha \in \Phi, k \in \mathbb{Z}$)

For $\lambda \in \Lambda$ let

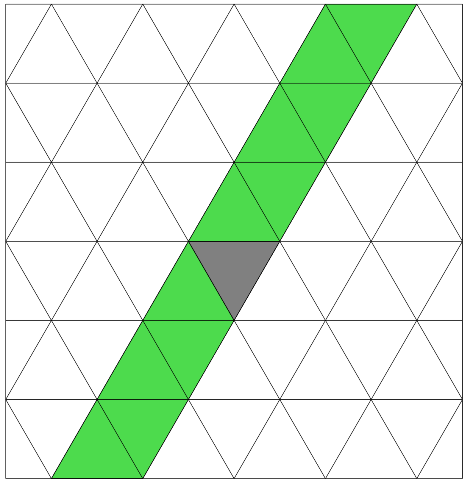
$\Phi^+_{J_\lambda}$

$$\mathcal{A}_\lambda = \{ x \in \mathbb{R}^n \mid 0 \leq \langle x, \alpha \rangle \leq 1 \quad \forall \alpha \in \Phi^+_\lambda \}$$

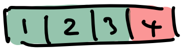
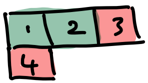
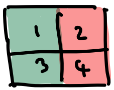
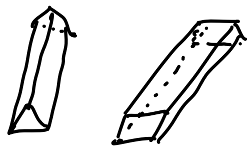
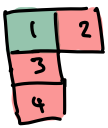
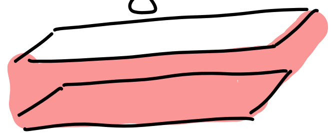
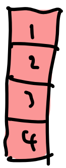
the "fundamental λ-alcove".

<u>Example</u> $\tilde{A}_2$

$$\lambda = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$$



Example $\tilde{A}_3$

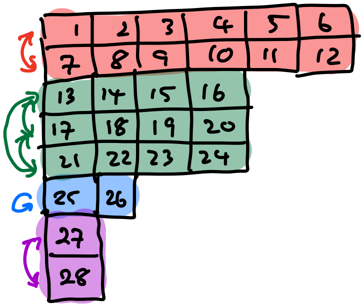
λ	Lower walls	Upper walls	$\mathcal{A}_\lambda$
	$H_{\alpha_1,0}$, $H_{\alpha_2,0}$, $H_{\alpha_3,0}$	$H_{\alpha_1+\alpha_2+\alpha_3,1}$	$\mathcal{A}_0$ (single alcove)
	$H_{\alpha_1,0}$, $H_{\alpha_2,0}$	$H_{\alpha_1+\alpha_2,1}$	"tubes"
	$H_{\alpha_1,0}$, $H_{\alpha_3,0}$	$H_{\alpha_1,1}$ $H_{\alpha_3,1}$	
	$H_{\alpha_1,0}$	$H_{\alpha_1,1}$	"layer" 
	$\emptyset$	$\emptyset$	$\mathbb{R}^3$

Symmetries and weights of $\mathcal{A}_\lambda$

Let

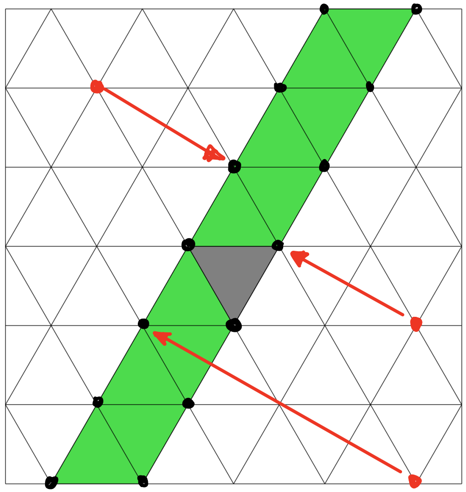
$$G_\lambda = \{ w \in W_0 \mid w\mathcal{A}_\lambda = \mathcal{A}_\lambda \}$$

Example. If

$\lambda =$ 

$$G_\lambda \simeq S_2 \times S_3 \times S_2$$

$$(A_1 \times A_2 \times A_1)$$



Let $Q_\lambda = \mathbb{Z}\text{-span } \Phi_\lambda$

$$T_\lambda = P/Q_\lambda$$

G_λ acts on T_λ.

$T_\lambda^+ =$ "dominant λ-weights"

$$\boxed{R[Z_\lambda] = R[T_\lambda]}$$

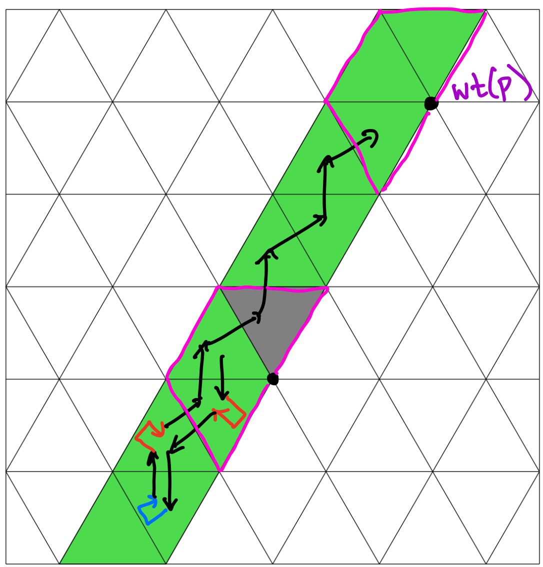
$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$ $\quad G_\lambda = \{e\}$ and $T_\lambda = \langle t_1, t_2 \mid 2t_1 + t_2 = 0 \rangle \simeq \mathbb{Z}$

$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline\end{array}$ $\quad G_\lambda \simeq A_1$ and $T_\lambda = \langle t_1, t_2 \mid 2t_1 + 2t_2 = 0 \rangle$

Combinatorial formula for $\pi_\lambda(T_w)$

Theorem (Path Formula)

$$[\pi_\lambda(T_w)]_{u,v} = \sum_{p \in \mathcal{P}_\lambda(\vec{w},u)_v} \left[(-q_r^{-1})^{b(p)} (q_r - q_r^{-1})^{f(p)}\right] z_\lambda^{wt(p)}$$



$\mathcal{P}_\lambda(\vec{w},u)_v$ = all λ-folded alcove paths of type $\vec{w}$ starting u, end v

$wt(p) \in T_\lambda$ weight

$f(p)$ = # folds of p [positively folded]

$b(p)$ = # bounces of p.

Calculating $c_\lambda(w)$

Let Γ_λ = left cell of Δ_λ containing $w_{\lambda'}$

Then $\Gamma_\lambda \cap \Gamma_\lambda^{-1} = \{m_\gamma \mid \gamma \in T_\lambda^+\}$.

Theorem (Leading matrices) For $\gamma \in T_\lambda^+$

$$c_\lambda(m_\gamma) = s_\gamma(z_\lambda) E_{1,1}$$

where $s_\gamma(z_\lambda) \in \mathbb{Z}[z_\lambda]^{G_\lambda}$ is a G_λ-Schur function

The proof consists of 3 main parts

1. λ-folded alcove paths $\longrightarrow$ find a single non-cancelling path of correct degree / weight
2. G_λ-invariance $\longleftarrow$ λ-relative Satake isomorphism
3. Determining Schur function $\longleftarrow$ Asymptotic Plancherel Formula

Conclusion

It is now a small step to conclude:

Theorem (Xi's Theorem)

$$H_\lambda^\infty \simeq \mathrm{Mat}_{N_\lambda,N_\lambda}\Big(\overbrace{\mathbb{Z}[z_\lambda]^{G_\lambda}}^{\simeq \mathrm{Rep}(F_\lambda)}\Big)$$