Random partitions at high temperature

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Based on joint works with **Maciej Dołęga**, and a joint paper with **Florent Benaych-Georges** and **Vadim Gorin**.

Plan of the talk

The Gaussian β-ensemble and semicircle distribution

LLN for random β-partitions at high temperature

The limiting measure: moment problem and Jacobi operators

Quantized γ-free convolution

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Gaussian Unitary Ensemble

The prob. measure on $\overline{\mathcal{W}_N} := \{a_1 \geqslant \cdots \geqslant a_N\} \subseteq \mathbb{R}^N$ with density

$$\boxed{\mathrm{Eigen}_N(a_1, \ldots, a_N) \propto \prod_{1 \leqslant i < j \leqslant N} (a_i - a_j)^2 \prod_{k=1}^{N} e^{-\frac{1}{2} a_k^2}}.$$

determines a random N-tuple of reals:

$$a_1 \geqslant \cdots \geqslant a_N.$$

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determines a random N-tuple of reals:

$$a_1 \geqslant \cdots \geqslant a_N.$$

This N-tuple is distributed like eigenvalues of the $N \times N$ complex Hermitian random matrix:

$$A_N = \frac{M_N + M_N^*}{2}, \quad M_N = [m_{ij}]_1^N, \quad m_{ij} = \mathcal{N}(0,1) + \mathrm{i} \cdot \mathcal{N}(0,1).$$

We call $\mathrm{Eigen}_N^{(2)}$ the Gaussian Unitary Ensemble (GUE).

Law of Large Numbers for GUE

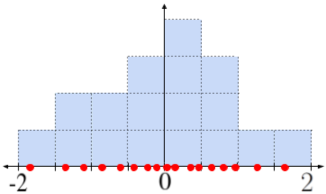
Consider the (random) empirical measures

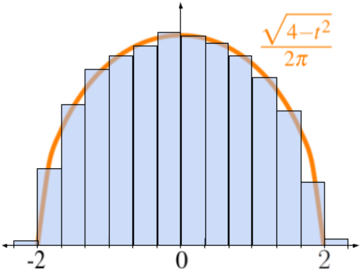
$$\mu_N := \frac{1}{N}\sum_{i=1}^{N}\delta_{\frac{a_i}{\sqrt{N}}}\ ,\quad \text{where } a_1 \geqslant \cdots \geqslant a_N \text{ is } \mathrm{Eigen}_N^{(2)}\text{-distributed.}$$

Theorem (Wigner '55)

The empirical measures μ_N converge weakly, in probability, to the semicircle distribution, with density

$$s(t) := 1_{\{-2\leqslant t\leqslant 2\}} \cdot \frac{\sqrt{4-t^2}}{2\pi}.$$





Moment method and multivariate Bessel functions

The typical proof of Wigner's theorem uses the moment method and reduces to finding the limits

$$\lim_{N\to\infty} \frac{1}{N^{k+1}} \cdot \mathbb{E}\Big[\mathrm{Tr}\big(A_N^{2k}\big)\Big] = \lim_{N\to\infty} \int_{\mathbb{R}} x^{2k} \mu_N(\mathrm{d}x), \quad \text{for all } k \geqslant 1.$$

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New ideas, due to [Bufetov–Gorin '15], employ the multivariate Bessel functions

$$B_{(a_1,\dots,a_N)}(x_1,\dots,x_N) := 1!\cdot 2! \cdots (N-1)! \cdot \frac{\det\left[e^{a_i x_j}\right]_{i,j=1}^N}{\prod_{i<j}(x_i - x_j)(a_i - a_j)},$$

s.t. $B_{(a_1,\dots,a_N)}(0^N) = 1$, and are diagonalized by differential operators

$$\mathcal{P}_k := \frac{1}{\prod_{i<j}(x_i - x_j)} \circ \sum_{i=1}^N \frac{\partial^k}{\partial x_i^k} \circ \prod_{i<j}(x_i - x_j), \quad k \geqslant 1,$$

namely,

$$\boxed{\mathcal{P}_k\Big(B_{(a_1,\dots,a_N)}(x_1,\dots,x_N)\Big) = \sum_{i=1}^N (a_i)^k \cdot B_{(a_1,\dots,a_N)}(x_1,\dots,x_N).}$$

Bessel generating functions

Main idea is to associate $\mathrm{Eigen}_N(a_1, \ldots, a_N) \mapsto F_N(x_1, \ldots, x_N)$, to GUE its Bessel generating function (a Fourier-type transform):

$$\boxed{F_N(x_1, \ldots, x_N) := \int B_{(a_1,\ldots,a_N)}(x_1, \ldots, x_N) \mathrm{Eigen}_N(a_1, \ldots, a_N) \mathrm{d}a_1 \ldots \mathrm{d}a_N.}$$

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The moments of empirical measures are exactly the "Taylor coeffs", i.e. first apply $\mathcal{P}_k$, and then find the constant term $\Big|_{x_1=\cdots=x_N=0}$:

$$\mathcal{P}_k F_N\Big|_{x_1=\cdots=x_N=0} = \mathbb{E}_{\mu_N}\left[\sum_{i=1}^{N} (a_i)^k\right].$$

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Upshot: The moments of μ_N can be accessed without matrices!

The difficulty now is to study limits of $\mathcal{P}_k$, applied to the BGF $F_N(x_1,\dots,x_N)$ $\big($which BTW equals $= e^{(x_1^2+\cdots+x_N^2)/2}\big)$, then set all $x_i=0$, and take the limit, as $N\to\infty$.

Dunkl operators

A new approach was started by [Benaych-Georges–C.–Gorin '22], who used instead the Dunkl differential-difference operators

$$\zeta_i := \frac{\partial}{\partial x_i} + \sum_{j\,:\,j\neq i} \frac{1}{x_i - x_j}(1 - s_{i,j}),$$

$$\widetilde{\mathcal{P}}_k := (\zeta_1)^k + \cdots + (\zeta_N)^k, \quad k \geqslant 1.$$

They also satisfy the equality

$$\widetilde{\mathcal{P}}_k\Big(B_{(a_1,\dots,a_N)}(x_1,\dots,x_N)\Big) = \sum_{i=1}^{N} (a_i)^k \cdot B_{(a_1,\dots,a_N)}(x_1,\dots,x_N)$$

but also admit a "β-generalization" $\cdots$

Gaussian Beta Ensemble

For general $\beta \geqslant 0$, we study the random N-tuples $a_1 \geqslant \cdots \geqslant a_N$ determined by the Gaussian β-ensemble (GβE):

$$\boxed{\mathrm{Eigen}_N^{(\beta)}(a_1, \cdots, a_N) \propto \prod_{1 \leqslant i < j \leqslant N} |a_i - a_j|^{\beta} \prod_{k=1}^{N} e^{-\frac{1}{2} a_k^2}}$$

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Motivations:

1. For $\beta = 1 \,\&\, 4$, it's the eigenvalue density of Gaussian Orthogonal Ensemble (GOE) & Gaussian Symplectic Ensemble (GSE).

2. $\mathrm{Eigen}_N^{(\beta)}$ is a Boltzmann distribution with logarithmic repulsion and the parameter β plays the role of inverse temperature.

3. Retains some integrability for all $\beta \geqslant 0$, e.g. normalization constant can be calculated from the Selberg integral.

Bessel generating functions

The relevant multivariate Bessel functions $B^{(\boldsymbol{\beta})}_{(a_1,\ldots,a_N)}(x_1,\ldots,x_N)$ are now abstract and defined by the β-Dunkl operators

$$\zeta_i^{(\boldsymbol{\beta})} := \frac{\partial}{\partial x_i} + \frac{\boldsymbol{\beta}}{2} \cdot \sum_{j:\, j \neq i} \frac{1}{x_i - x_j}(1 - s_{i,j}),$$

$$\mathcal{P}_k^{(\boldsymbol{\beta})} := \left(\zeta_1^{(\boldsymbol{\beta})}\right)^k + \cdots + \left(\zeta_N^{(\boldsymbol{\beta})}\right)^k, \quad k \geqslant 1,$$

and eigenfunction relations

$$\mathcal{P}_k^{(\boldsymbol{\beta})}\left(B^{(\boldsymbol{\beta})}_{(a_1,\ldots,a_N)}(x_1,\ldots,x_N)\right) = \sum_{i=1}^{N} (a_i)^k \cdot B^{(\boldsymbol{\beta})}_{(a_1,\ldots,a_N)}(x_1,\ldots,x_N)$$

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The relevant Fourier transform is now

$$\boxed{F_N^{(\beta)}(x_1,\dots,x_N) := \int B^{(\beta)}_{(a_1,\dots,a_N)}(x_1,\dots,x_N)\mathrm{Eigen}_N^{(\beta)}(a_1,\dots,a_N)da_1\dots da_N,}$$

and still satisfies: $$\mathcal{P}_k^{(\beta)} F_N^{(\beta)}\Big|_{x_1=\dots=x_N=0} = \mathbb{E}_{\mu_N}\left[\sum_{i=1}^{N} a_i^k\right].$$

LLN for GβE eigenvalues at fixed temperature

Nothing changes if $\beta > 0$ is fixed: as $N \to \infty$, then

$$\mu_N := \frac{1}{N} \sum_{i=1}^{N} \delta_{\frac{a_i}{\sqrt{N}}}, \text{ where } a_1 \geqslant \cdots \geqslant a_N \text{ is } \mathrm{Eigen}_N^{(\beta)}\text{-distributed,}$$

converge weakly, in probability, to a semicircle distribution.

In the extreme $\beta = 0$ case, the interaction $\prod_{i<j} |a_i - a_j|^{\beta}$ vanishes, and we get Gaussian distribution as the limit of empirical measures.

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Given $\gamma \in (0, \infty)$, we were interested in the crossover **high temperature regime**:

$$N \to \infty, \quad \beta \to 0^+, \quad \frac{N\beta}{2} \to \gamma,$$

hoping: when $\gamma \to \infty$, get semicircle distribution;
when $\gamma \to 0^+$, get Gaussian distribution.

LLN for GβE eigenvalues at fixed temperature

Theorem (Duy, Shirai '15 & Benaych-Georges, C, Gorin '22)

Consider the empirical measures

$$\mu_{N,\beta} := \frac{1}{N}\sum_{i=1}^{N}\delta_{a_i}, \quad \text{where } a_1 \geqslant \cdots \geqslant a_N \text{ is } \mathrm{Eigen}_N^{(\beta)}\text{-distributed.}$$

In the limit: $\quad N \to \infty, \quad \beta \to 0^+, \quad \dfrac{N\beta}{2} \to \gamma \in (0,\infty),$

the measures $\mu_{N,\beta}$ converge weakly, in probability, to certain probability measure $\mu^{(\gamma)}$.

The density of $\mu^{(\gamma)}$ is complicated, but explicit, and contained in [Allez–Bouchaud–Guionnet '12].

Global asymptotics of GβE eigenvalues at high temp

Theorem (Duy, Shirai '15 & Benaych-Georges, C, Gorin '22)

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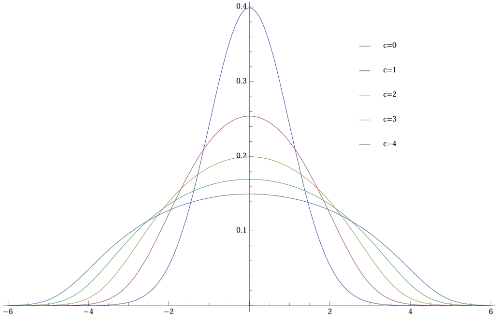
In the limit: $\quad N \to \infty, \quad \beta \to 0^+, \quad \frac{N\beta}{2} \to \gamma \in (0,\infty),$

we have $\mu_{N,\beta} \to \mu^{(\gamma)}$ *weakly, in probability.*



Density of $\mu^{(\gamma)}$ for various γ (courtesy of [Allez–Bouchaud–Guionnet '12])

Moments of the limiting measure $\mu^{(\gamma)}$

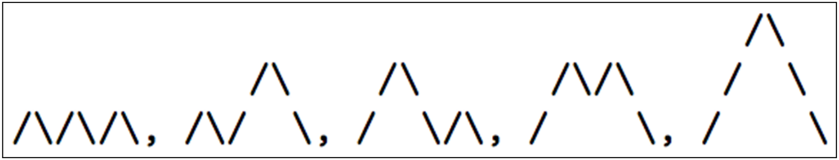
As a result of the moment method, we got new moment formulas:

Theorem (Benaych-Georges – Cuenca – Gorin '22)

The limiting measure $\mu^{(\gamma)}$ is uniquely determined by its moments:

$$\int_{-\infty}^{\infty} x^k \mu^{(\gamma)}(dx) = \sum_{\text{Dyck paths } \Gamma \text{ of length } k} weight(\Gamma),$$

where: $weight(\Gamma) := \prod_{j \geqslant 1} (j+\gamma)^{\#\text{down steps from height } j}.$



$(1+\gamma)^3$, $(1+\gamma)^2(2+\gamma)$, $(1+\gamma)^2(2+\gamma)$, $(1+\gamma)(2+\gamma)^2$, $(1+\gamma)(2+\gamma)(3+\gamma)$

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LLN for random β-partitions at high temperature

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Discrete Beta Ensembles

Now study discrete random partitions. There are many definitions.

We are motivated by the **discrete β-ensembles**, due to [Borodin–Gorin–Guionnet '17], on $\overline{\mathcal{W}_{N,\mathbb{Z}}} := \{(\lambda_1 \geqslant \cdots \geqslant \lambda_N) \in \mathbb{Z}^N\}$:

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- the parameter $\boxed{\theta = \dfrac{\beta}{2}}$ is more natural;
- the shifted coordinates $\boxed{\ell_i := \lambda_i - (i-1)\theta}$ are more natural, so that $\ell_1 > \cdots > \ell_N$;
- they considered probability measures

$$\mathbb{P}_N\big(\ell_1 > \cdots > \ell_N\big) \propto \prod_{1 \leqslant i < j \leqslant N} \frac{\Gamma(\ell_i - \ell_j + 1)\Gamma(\ell_i - \ell_j + \theta)}{\Gamma(\ell_i - \ell_j)\Gamma(\ell_i - \ell_j + 1 - \theta)} \prod_{k=1}^{N} w(\ell_k; N).$$

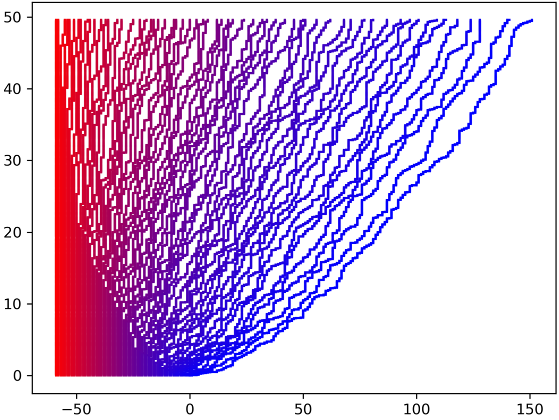
- Among other things, they proved the LLN for $\mu_N = \frac{1}{N}\sum_{i=1}^{N} \delta_{\ell_i}$.

Discrete Dyson Brownian Motion

Another motivation was [Gorin–Shkolnikov '15], who defined a continuous-time, discrete-space, one-parameter θ-dependent (growing) Markov chain

$$\big(\ell_1(t) > \cdots > \ell_N(t)\big), \quad t \geqslant 0,$$

that should be regarded as the discrete Dyson Brownian motion.



Initial condition: $\lambda(0) = (0, 0, \ldots, 0)$, or $\ell_i(0) = -(i-1)\theta$.

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- [discrete $\to$ continuous space limit] gives Dyson Brownian motion;
- it is a random evolution of N non-intersecting particles (not a Doob h-transform, unless $\theta = 1$);
- if started at $\lambda(0) = (0, \cdots, 0)$, equivalently $\ell_i(0) = -(i-1)\theta$, then at time t, you get the discrete β-ensemble with

$$w(x; N) = \frac{t^x}{\Gamma(x + (N-1)\theta + 1)}, \quad x \geqslant 0.$$

Jack generating functions

More generally, we consider measures with nice (analytic near 1^N) Jack generating functions

$$\boxed{F^{(\theta)}_{\mathbb{P}_N}(x_1,\ldots,x_N) := \sum_{\lambda} \mathbb{P}_N(\lambda)\,\frac{J^{(\theta)}_\lambda(x_1,\ldots,x_N)}{J^{(\theta)}_\lambda(1^N)},}$$

where $J^{(\theta)}_\lambda(x_1,\ldots,x_N)$ are Jack symmetric polynomials, defined by

$$\left(\xi_1^k+\cdots+\xi_N^k\right)J^{(\theta)}_\lambda(x_1,\ldots,x_N) = \sum_{i=1}^{N}(\ell_i)^k\cdot J^{(\theta)}_\lambda(x_1,\ldots,x_N), \quad k\geqslant 1,$$

where the Cherednik operators $\xi_1,\ldots,\xi_N$ are

$$\xi_i := \theta(1-i)+x_i\frac{\partial}{\partial x_i}+\theta\sum_{j=1}^{i-1}\frac{x_i}{x_i-x_j}(1-s_{i,j})+\theta\sum_{j=i+1}^{N}\frac{x_j}{x_i-x_j}(1-s_{i,j}).$$

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More generally, we consider measures with nice (analytic near 1^N) Jack generating functions

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A variation was used for LLN/CLT of models at fixed temperature, by [Huang '20]. For high temperature, we proved $\cdots$

Law of Large Numbers for random θ-partitions

Theorem [C.–Dołęga '25].
Let $\{\mathbb{P}_N\}_{N\geqslant 1}$ be measures on partitions $\lambda_1 \geqslant \cdots \geqslant \lambda_N$, with empirical measures $\mu_N := \frac{1}{N}\sum_{i=1}^N \delta_{\ell_i}$, and $\ell_i := \lambda_i + \theta(i-1)$.

Assume that the JGF's $\left\{F_N^{(\theta)}(x_1,\dots,x_N)\right\}_{N\geqslant 1}$ satisfy:

- $\displaystyle\lim_{\substack{N\to\infty \\ N\theta\to\gamma}} \frac{1}{(\ell-1)!}\frac{\partial^\ell}{\partial x_1^\ell} F_N^{(\theta)}\Big|_{x_1=\cdots=x_N=0} = \kappa_\ell$, for all $\ell \geqslant 1$.
- $\displaystyle\lim_{\substack{N\to\infty \\ N\theta\to\gamma}} \frac{\partial^r}{\partial x_{i_1}\cdots\partial x_{i_r}} F_N^{(\theta)}\Big|_{x_1=\cdots=x_N=0} = 0$, for all mixed derivatives.

Then there is a prob. measure $\mu^{(\gamma)}$ with finite moments $m_1, m_2, \dots$

s.t. $\boxed{\lim_{N\to\infty} \mu_N = \mu^{(\gamma)}}$ in the sense of moments, in probability.

Law of Large Numbers for random θ-partitions

Theorem [C.–Dołęga '25]. LLN for empirical measures if

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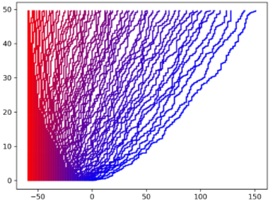
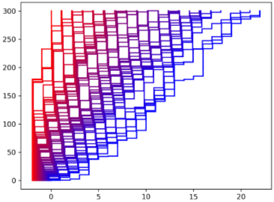
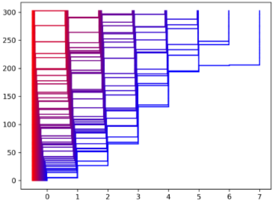
Example: Fixed-time t distribution of GS process: $\boxed{\kappa_\ell = \delta_{\ell,1}\cdot t}$



(A) $\theta = 1, N = 60$



(B) $\theta = \frac{2}{N}, N = 60$



(C) $\theta = \frac{1}{2N}, N = 60$

Law of Large Numbers for random θ-partitions

Theorem [C.–Dołęga '25]. LLN for empirical measures if

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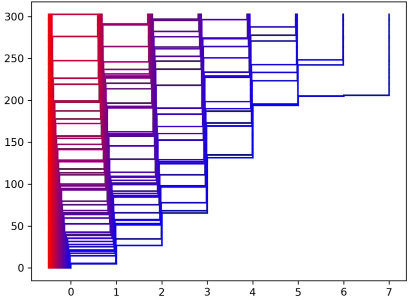

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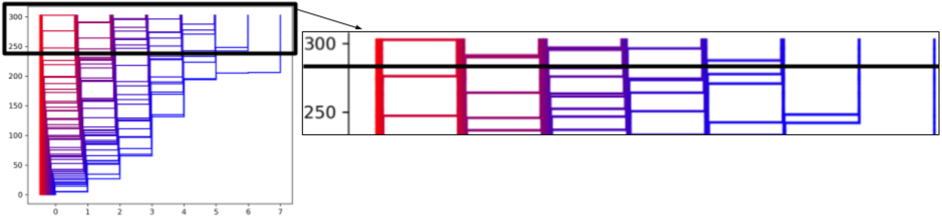

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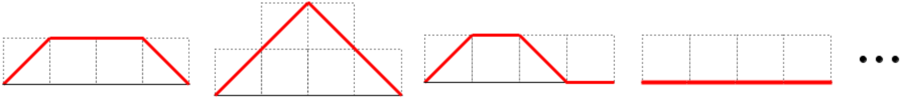
Moment problem

Recall that for fixed-time distribution of GS process, started from $\lambda(0) = (0, 0, \cdots, 0)$, we have $\mu_N(t) \to \mu_{\text{Planch}}^{(\gamma,t)}$.

Theorem [C.–Dołęga '25]. The prob. measure $\mu_{\text{Planch}}^{(\gamma,t)}$ is uniquely determined by its moments: for $t = \gamma$, and all $\ell \geqslant 1$:

$$\boxed{\begin{aligned}\int_{\mathbb{R}} x^\ell \mu_{\text{Planch}}^{(\gamma,\gamma)}(dx) = \sum_{\Gamma \in \text{Motzkin}(\ell)} & \frac{\gamma^{\#\text{up steps} + \#\text{hor steps at height } 0}}{1 + \#\text{hor steps at height } 0} \\ & \cdot \prod_{j \geqslant 1} (j+\gamma)^{\#\text{hor steps at height } j + \#\text{down steps from height } j}.\end{aligned}}$$

Note that: weight$(\Gamma) \neq \prod_{e \in E(\Gamma)} f(e)$!



$\gamma(1+\gamma)^3$, $\gamma^2(2+\gamma)(1+\gamma)$, $\frac{1}{2}\gamma^2(1+\gamma)^2$, $\frac{1}{5}\gamma^4$, $\ldots$

Inverse transform problem

Recall that for fixed-time distribution of GS process, started from $\lambda(0) = (0, 0, \cdots, 0)$, we have $\mu_N(t) \to \mu_{\mathrm{Planch}}^{(\gamma,t)}$.

Theorem [C.–Dołęga '25]. As formal power series in z^{-1}:

$$\sum_{n \geqslant 0} \frac{\gamma^{\uparrow n}(-t)^n}{z^{\uparrow n} n!} = \exp\left(\gamma \cdot \widetilde{\mathcal{L}}\left\{ \int_{\mathbb{R}} e^{-xa} \mu_{\mathrm{Planch}}^{(\gamma,t)}(da) - \frac{e^{\gamma x} - 1}{\gamma x} \right\}(z)\right) \quad (*)$$

where $z^{\uparrow n} := z(z+1)\cdots(z+n-1)$, and $\widetilde{\mathcal{L}}$ is the formal LT:

$$\widetilde{\mathcal{L}}\left\{ \sum_{n \geqslant 0} \frac{s_n}{n!} x^n \right\}(z) := \sum_{n \geqslant 0} s_n z^{-n-1}.$$

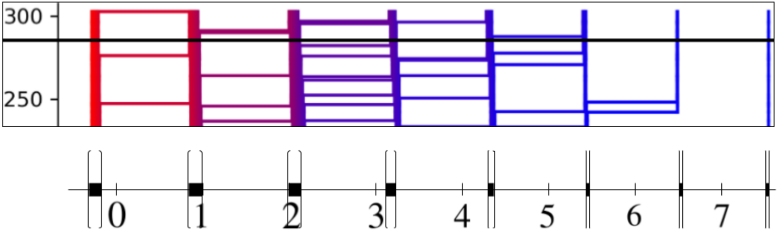
Jacobi operators

LHS of (*) is the characteristic function of certain Jacobi operator

$$\mathcal{J}^{(\gamma,t)}_{\text{Planch}} = \begin{bmatrix} a_1(\gamma,t) & b_1(\gamma,t) & 0 & \cdots \\ b_1(\gamma,t) & a_2(\gamma,t) & b_2(\gamma,t) & \\ 0 & b_2(\gamma,t) & a_3(\gamma,t) & \\ \vdots & & & \ddots \end{bmatrix},$$

i.e. zeroes of the LHS of (*) $\approx$ eigenvalues of $\mathcal{J}^{(\gamma,t)}_{\text{Planch}}$.

The exact description of $\mu^{(\gamma,t)}_{\text{Planch}}$ in terms of eigenvalues of $\mathcal{J}^{(\gamma,t)}_{\text{Planch}}$ will be discussed in a future paper (ongoing work with Dołega).



Depiction of the support of $\mu^{(\gamma,t)}_{\text{Planch}}$.

Plan of the talk

The Gaussian β-ensemble and semicircle distribution

LLN for random β-partitions at high temperature

The limiting measure: moment problem and Jacobi operators

Quantized γ-free convolution

Discrete DBM with arbitrary initial condition

Assume that we perform the Markov evolution of Gorin–Shkolnikov for arbitrary initial conditions

$$\ell^{(N)}(0) := \left(\ell_1^{(N)}(0) > \ell_2^{(N)}(0) > \cdots > \ell_N^{(N)}(0)\right),$$

that satisfy

$$\frac{1}{N}\sum_{i=1}^{N}\delta_{\ell_i^{(N)}(0)} \xrightarrow{N\to\infty} \nu.$$

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Theorem [C.–Dołęga '25]. Let $\ell^{(N)}(t) = \big(\ell_1^{(N)}(t) > \ldots > \ell_N^{(N)}(t)\big)$ be the Markov chain at time $t > 0$. There exists a prob. measure μ s.t.

$$\lim_{\substack{N\to\infty \\ N\theta\to\gamma}} \frac{1}{N}\sum_{i=1}^{N} \delta_{\ell_i^{(N)}(t)} = \mu.$$

Moreover, μ is uniquely determined by its moments, or equivalently:

$$\boxed{\kappa_n^{(\gamma)}[\mu] = \kappa_n^{(\gamma)}[\nu] + \kappa_n^{(\gamma)}\big[\mu_{\text{Planch}}^{(\gamma,t)}\big], \text{ for all } n \geqslant 1.}$$

Quantized γ-free convolution

Question

Given two probability measures μ, ν *with finite* $\kappa_n^{(\gamma)}[\mu]$, $\kappa_n^{(\gamma)}[\nu]$, *does there exist a third probability measure* $\mu \boxplus^{(\gamma)} \nu$, *such that*

$$\kappa_n^{(\gamma)}\left[\mu \boxplus^{(\gamma)} \nu\right] = \kappa_n^{(\gamma)}[\mu] + \kappa_n^{(\gamma)}[\nu], \text{ for all } n \geqslant 1?$$

Our theorem answers affirmatively when $\mu = \mu_{\mathrm{Planch}}^{(\gamma,t)}$, and ν is of the form $\nu = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \delta_{\ell_i^{(N)}(0)}$.

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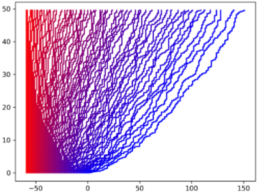
Conjecture

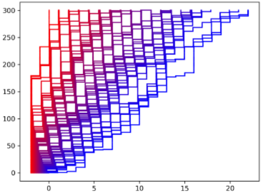
The answer to the question is always YES.

The question is related to the conjecture of [Stanley '89] on the integrality/positivity of Littlewood-Richardson coeffs of Jack polys.

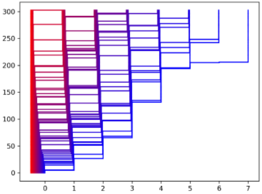
1-parameter γ-generalization of the (quantized) free convolution? ([Voiculescu '92], [Speicher '94], [Bufetov–Gorin '15]).



(A) $\theta = 1, N = 60$



(B) $\theta = \frac{2}{N}, N = 60$



(C) $\theta = \frac{1}{2N}, N = 60$

Thank you for your attention!